Interpolation of entropy moduli between Hilbert spaces

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Definition

The k-th entropy number $\varepsilon_k(T)$ of $T \in L(X, Y)$ is defined by

$$\varepsilon_k(T) := \inf\left\{ \varepsilon > 0 \;:\; T(U_X) \subset \bigcup_{i=1}^{k} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

Entropy numbers and moduli

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- The entropy moduli

$$g_s(T) := \inf_{k \in \mathbb{N}} k^{1/(2s)} \varepsilon_k(T), \quad s \in (0, \infty)$$

- Multiplicativity:

$$g_s(RS) \leqslant g_s(R)\, g_s(S) \ \text{ for } \ S \in L(X, Z), R \in L(Z, Y)$$

Carl-Triebel's inequality

- Let $\{\lambda_n(T)\}_{n=1}^{\infty}$ be an eigenvalue sequence of $T \in K(X)$ on a complex Banach space X.

- [Carl-Triebel, 1980]

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant g_n(T) = \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

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A classical Riesz theory

- The essential spectral radius is given by

$$r_{ess}(T) := \sup_{\lambda \in \sigma_{ess}(T)} |\lambda| = \lim_{m \to \infty} \|T^m\|_{ess}^{1/m}$$

- T is a Riesz operator $\iff r_{ess}(T) = 0$. Every power compact operator is a Riesz operator.

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- T is a Riesz operator $\iff r_{ess}(T) = 0$. Every power compact operator is a Riesz operator.
- The Riesz part of the spectrum

$$\Lambda(T) := \{\, \lambda \in \sigma(T) \ : \ |\lambda| > r_{ess}(T) \,\}$$

is at most countable and consists of isolated eigenvalues of finite algebraic multiplicity.

Eigenvalue sequence

- The Riesz part of the spectrum

$$\Lambda(T) := \{\, \lambda \in \sigma(T) \;:\; |\lambda| > r_{ess}(T) \,\}$$

We assign an eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ for an operator $T \in L(X)$ from the elements of the set $\Lambda(T) \cup \{r_{ess}(T)\}$ as follows:

- The eigenvalues are arranged in an order of non-increasing absolute values.
- Every eigenvalue $\lambda \in \Lambda(T)$ is counted according to its algebraic multiplicity.
- If T possesses less than n eigenvalues λ with $|\lambda| > r_{ess}(T)$, we let

$$\lambda_n(T) = \lambda_{n+1}(T) = \ldots = r_{ess}(T)$$

Banach couple

- We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

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For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1}\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1}\right\}$$

Interpolation functor

- By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that

$$T|_{A_0} \in L(A_0, B_0) \text{ and } T|_{A_1} \in L(A_1, B_1)$$

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Definition

By an interpolation functor we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$

- $A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1$ for any $\vec{A} \in \vec{\mathcal{B}}$
- $T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B})$ for any $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and $T\colon \vec{A} \to \vec{B}$

Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \max\left\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\right\}$$

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If in addition there exists $\theta \in (0,1)$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \|T\|_{A_0\to B_0}^{1-\theta} \|T\|_{A_1\to B_1}^{\theta},$$

then $\mathcal{F}$ is called of exponential type of θ.

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are of exponential type of θ.

Recollect...

- The k-th entropy number $\varepsilon_k(T)$ of $T \in L(X, Y)$

$$\varepsilon_k(T) := \inf\left\{ \varepsilon > 0 \;:\; T(U_X) \subset \bigcup_{i=1}^{k} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

- The entropy moduli

$$g_s(T) := \inf_{k \in \mathbb{N}} k^{1/(2s)} \varepsilon_k(T), \quad s \in (0, \infty)$$

- [Carl-Triebel, 1980], [Makai-Zemánek, 1982]

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} \leqslant g_n(T) = \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

One-sided interpolation results

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples and $\theta \in (0,1)$.

- If X belongs to the class $\mathcal{C}_K(\theta; \vec{A})$ and $B := B_0 = B_1$, then

$$g_s(T\colon X \to B) \leqslant C\, g_{s(1-\theta)}(T\colon A_0 \to B)^{1-\theta}\, g_{s\theta}(T\colon A_1 \to B)^{\theta}\,.$$

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- If $A := A_0 = A_1$ and Y belongs to the class $\mathcal{C}_J(\theta; \vec{B})$, then

  $$g_s(T\colon A \to Y) \leqslant C\, g_{s(1-\theta)}(T\colon A \to B_0)^{1-\theta}\, g_{s\theta}(T\colon A \to B_1)^{\theta}\,.$$

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are members of $\mathcal{C}_K(\theta; \cdot)$ and $\mathcal{C}_J(\theta; \cdot)$.

Two-sided interpolation of entropy moduli ?

A delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ.
Does there exist a constant C such that for any $T\colon \vec{A} \to \vec{B}$

$$g_s\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, g_s(T\colon A_0 \to B_0)^{1-\theta}\, g_s(T\colon A_1 \to B_1)^{\theta}$$

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- [Edmunds-Netrusov, 2011]
  The entropy numbers do not interpolate well at least in the situation of the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$.

Two-sided interpolation of entropy moduli ?

A delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ.
There cannot be any constant C such that for all $T\colon \vec{A} \to \vec{B}$

$$g_s\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, g_s(T\colon A_0 \to B_0)^{1-\theta}\, g_s(T\colon A_1 \to B_1)^{\theta}$$

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- [Edmunds-Netrusov, 2011]
  The entropy numbers do not interpolate well at least in the situation of the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$.
- [Mastyło-Szwedek]
  We transfer this example to entropy moduli.

Two-sided interpolation of entropy moduli

Interpolation of entropy moduli between Hilbert spaces

Theorem [Szwedek, 2015]

There exists a constant $C > 0$, such that

- *for all couples $\vec{H} = (H_0, H_1)$ and $\vec{K} = (K_0, K_1)$ of complex Hilbert spaces, and*
- *for every operator $T \in L(\vec{H}, \vec{K})$ and every $\theta \in (0,1)$*

$$g_n\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant C\, g_n(T\colon H_0 \to K_0)^{1-\theta}\, g_n(T\colon H_1 \to K_1)^{\theta} \ .$$

Hilbert spaces, approximation numbers and eigenvalues

- The s-numbers of operators between Hilbert spaces coincide.
- Approximation numbers of $T \in L(H, K)$

$$a_n(T) = \inf\{\|T - TP\| \ : P \in L(H) \text{ orthog. proj. with } \text{rank}\, P < n\}$$

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$$a_n(T) = \inf\{\|T - TP\| \ : P \in L(H) \text{ orthog. proj. with } \operatorname{rank} P < n\}$$

- $N \in L(H)$ is normal if $N^*N = NN^*$. [Szwedek, 2015] If N is normal, then
$$|\lambda_n(N)| = a_n(N)\,.$$
- $|T| := D$ such that $T^*T = D^2$ where $T^*\colon K \to H$.

$$\lambda_n(|T|) = a_n(|T|) = a_n(T\colon H \to K) = a_n(T^*\colon K \to H)\,.$$

Hilbert spaces, approximation numbers and entropy moduli

Definition

The n-th entropy number $\varepsilon_n(T)$ of $T \in L(H,K)$ is defined by

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \;:\; T(U_H) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_K\}, \quad y_i \in K \right\}$$

- The n-th entropy moduli

$$g_n(T) := \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

- $$\Big(\prod_{i=1}^{n} a_i(T)\Big)^{1/n} \asymp g_n(T)$$

Geometric interpolation between Hilbert spaces (1)

Let $\vec{H} = (H_0, H_1)$ be a regular couple of Hilbert spaces.

- We define interpolation spaces using powers of a positive operator [Donoghue, 1967], [M^cCarthy, 1992]

Geometric interpolation between Hilbert spaces (1)

Let $\vec{H} = (H_0, H_1)$ be a regular couple of Hilbert spaces.

- We define interpolation spaces using powers of a positive operator [Donoghue, 1967], [McCarthy, 1992]
- The inner product for H_1 is a Hermitian form on $H_0 \cap H_1$, so there exists a densely defined, positive injective operator A on H_0 satisfying

$$\langle \xi, \eta \rangle_1 = \langle A^{1/2}\xi, A^{1/2}\eta \rangle_0 \text{ for all } \xi, \eta \in H_0 \cap H_1.$$

- $H_0 \cap H_1$ is contained in both Dom $A^{1/2}$ and Ran $A^{1/2}$.
- A is bounded if and only if H_0 is embedded in H_1.

Geometric interpolation between Hilbert spaces (2)

Let $\vec{H} = (H_0, H_1)$ be a regular couple of Hilbert spaces.

Definition of H_θ

For fixed $\theta \in (0,1)$, we define a new inner product on $H_0 \cap H_1$ by

$$\langle \xi, \eta \rangle_\theta = \langle A^{\theta/2}\xi, A^{\theta/2}\eta \rangle_0 .$$

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- $H_0 \cap H_1$ is contained in both $\mathrm{Dom}\, A^{\theta/2}$ and $\mathrm{Ran}\, A^{\theta/2}$.
- H_θ - the closure of $H_0 \cap H_1$, with respect to the norm given by the inner product $\langle \cdot, \cdot \rangle_\theta$.
- $$H_\theta \cong [H_0, H_1]_\theta.$$

Unitary equivalence of entropy moduli

Lemma - "unitary equivalence" [Szwedek, 2015]

Let $\vec{H}$ and $\vec{K}$ be regular couples of Hilbert spaces. Assume that

- *A is a positive operator on H_0 that generate the H_1 inner product, and*
- *B is a positive operator on K_0 that generate the K_1 inner product,*
- *$T \in L(\vec{H}, \vec{K})$ and $\theta \in [0, 1]$.*

Then

$$g_n(T\colon H_\theta \to K_\theta) = g_n\left(B^{\theta/2} T A^{-\theta/2}\colon H_0 \to K_0\right)$$

Theorem [Szwedek, 2015] (reduced form)

There exists a constant $C > 0$, such that

- *for all couples $\vec{H} = (H_0, H_1)$ and $\vec{K} = (K_0, K_1)$ of complex Hilbert spaces, and*
- *for every operator $T \in L(\vec{H}, \vec{K})$ and every $\theta \in (0, 1)$*

$$A_n\big(T\colon H_\theta \to K_\theta\big) \leqslant C\, A_n\big(T\colon H_0 \to K_0\big)^{1-\theta} A_n\big(T\colon H_1 \to K_1\big)^{\theta},$$

where $A_n(T)$ denotes $\big(\prod_{i=1}^{n} a_i(T)\big)^{1/n} \asymp g_n(T)$.

Sketch of the proof of the main theorem (1)

- "unitary equivalence" of entropy moduli

$$A_k(T\colon H_\theta \to K_\theta) \asymp A_k\big(R_\theta\colon H_0 \to K_0\big)$$

$$A_n(T) := \big(\prod_{i=1}^{n} a_i(T)\big)^{1/n} \text{ and } R_\theta := B^{\theta/2} T A^{-\theta/2} \text{ for } \theta \in (0,1)$$

- $\theta = 1/2$

$$\begin{aligned} A_k\big(R_{1/2}\big) &= \left(\prod_{i=1}^{n} a_i\big(R_{1/2}\big)\right)^{1/n} = \left(\prod_{i=1}^{n} \big|\lambda_i\big(|R_{1/2}|\big)\big|\right)^{1/n} \\ &= \left(\prod_{i=1}^{n} \Big|\lambda_i\Big(\big|R_{1/2}\big|^2\Big)\Big|\right)^{1/2n} = \left(\prod_{i=1}^{n} \Big|\lambda_i\Big(R_{1/2}^* R_{1/2}\Big)\Big|\right)^{1/2n} \\ &= \left(\prod_{i=1}^{n} \Big|\lambda_i\Big(A^{1/4} R_{1/2}^* R_{1/2} A^{-1/4}\Big)\Big|\right)^{1/2n} \end{aligned}$$

Sketch of the proof of the main theorem (2)

- $R_\theta := B^{\theta/2} T A^{-\theta/2}$ and $R_\theta^* := A^{-\theta/2} T^* B^{\theta/2}$

$$\begin{aligned}
A_k(R_{1/2}) &= \left(\prod_{i=1}^{n} \left| \lambda_i \left(\left| R_{1/2} \right|^2 \right) \right| \right)^{1/2n} = \left(\prod_{i=1}^{n} \left| \lambda_i \left(R_{1/2}^* R_{1/2} \right) \right| \right)^{1/2n} \\
&= \left(\prod_{i=1}^{n} \left| \lambda_i \left(A^{1/4} R_{1/2}^* R_{1/2} A^{-1/4} \right) \right| \right)^{1/2n} \\
&\leqslant g_k \left(A^{1/4} R_{1/2}^* R_{1/2} A^{-1/4} \right)^{1/2} \\
&= g_k \left(T^* B^{1/2} T A^{-1/2} \right)^{1/2} \\
&\leqslant g_k \left(T^* \right)^{1/2} g_k \left(B^{1/2} T A^{-1/2} \right)^{1/2} \\
&= g_k \left(T \right)^{1/2} g_k \left(R_1 \right)^{1/2} \asymp A_k(R_0)^{1/2} A_k(R_1)^{1/2}
\end{aligned}$$

Sketch of the proof of the main theorem (3)

- $\theta = 1/2$:

$$A_k\left(R_{1/2}\right) \leqslant C\, A_k(R_0)^{1/2}\, A_k(R_1)^{1/2}$$

- $\theta = 1/4$: interpolating between R_0 and $R_{1/2}$ gives

$$A_k\left(R_{1/4}\right) \leqslant C^{3/2} A_k(R_0)^{1/2}\, A_k\left(R_{1/2}\right)^{1/2}$$

- $\theta = 3/4$: interpolating between $R_{1/2}$ and R_1 gives

$$A_k\left(R_{3/4}\right) \leqslant C^{3/2} A_k\left(R_{1/2}\right)^{1/2} A_k(R_1)^{1/2}$$

- Theorem holds for any dyadic rational in $[0, 1]$

$$A_k(R_\theta \colon H_0 \to K_0) \leqslant C^2 A_k(R_0 \colon H_0 \to K_0)^{1-\theta}\, A_k(R_1 \colon H_0 \to K_0)^{\theta}$$