Banach spaces with prescribed ultrapowers

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Basic definitions

Definition

Let $(E_i)_{i\in I}$ be a family of Banach spaces indexed by I and $\mathcal{U}$ be an utrafilter over the index set I. The *ultraproduct* $\prod_{i,\mathcal{U}} E_i$ (or simply $\prod_{\mathcal{U}} X_i$) is defined as the quotient Banach space

$$\ell_\infty(E_i; i \in I)/c_{0,\mathcal{U}}(E_i; i \in I)$$

with $\ell_\infty(E_i; i \in I)$= space of bounded families in $\prod_{i\in I} E_i$ with sup-norm and $c_{0,\mathcal{U}}(E_i; i \in I)$= the subspace of families $\mathcal{U}$-converging to zero.

If $E_i = E$ for all $i \in I$ then we speak of an *ultrapower* of E, denoted by $E_{\mathcal{U}}$.

Notation

If $(x_i) \in \ell_\infty(E_i; i \in I)$ denote by $[x_i]_{\mathcal{U}}$ the element it defines in $\prod_{i,\mathcal{U}} E_i$.

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Some more basics

A Recall

Recall that the $\mathcal{U}$-limit ℓ of a family of points (t_i) in a topological space T, if it exists, is characterized by

$$\forall V \text{ neighborhood of } \ell, \{i \in I : t_i \in V\} \in \mathcal{U}$$

and denoted by $\lim_{i,\mathcal{U}} t_i$. It always exists if the space T is compact.

A formula for the ultraproduct norm

$$\|[x_i]_{\mathcal{U}}\| = \lim_{i,\mathcal{U}} \|x_i\|$$

Ultraproducts of operators

If $T_i : X_i \to Y_I$ is a uniformly bounded family of linear operators, one may define unambiguously $\tilde{T} : \prod_{i,\mathcal{U}} X_i \to \prod_{i,\mathcal{U}} Y_i$ by $\tilde{T}[x_i]_{\mathcal{U}} = [T_i x_i]_{\mathcal{U}}$.
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Canonical embeddings

The diagonal embedding

For X a Banach space, $\mathcal{U}$ an ultrafilter, set $D_X x = [x]_{\mathcal{U}}$ for $x \in X$. This defines a linear isometric embedding $D_X : X \to X_{\mathcal{U}}$.

An ultraproduct version

Let X a Banach space and $(E_i)_{i \in I}$ an upwards directed net of closed linear subspaces of X ($\forall i, j \, \exists k$ such that $E_k \supset E_i + E_j$) with dense union E_0. Let's call *adapted* an ultrafilter $\mathcal{U}$ on I if $\mathcal{U} \ni \{j \in I : E_j \supset E_i\}$ for all i. Denote by u_i be the inclusion $E_i \hookrightarrow X$. For each adapted $\mathcal{U}$ there is a unique map $D : X \to \prod_{i,\mathcal{U}} E_i$ such that $(\prod_{i,\mathcal{U}} u_i) \circ D = D_X$, and D is an isometric linear embedding.

Let $u_{i,j}$ be the inclusion $E_i \hookrightarrow E_j$, if $E_i \subset E_j$, $u_{i,j} = 0$ if not. Then $D_i : E_i \hookrightarrow \prod_{j,\mathcal{U}} E_j$ by $D_i x = [u_{i,j}(x)]_{j,\mathcal{U}}$ is a linear isometric embedding, the D_i are compatible and define together $D_0 : E_0 \hookrightarrow \prod_{j,\mathcal{U}} E_i$, that extends by density to the whole of X. Unicity of D because $\prod_{i,\mathcal{U}} u_i$ is isometric.

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Lattice ultraproducts

These concepts extend easily to other categories of *normed structures*, i. e. Banach spaces with additional structures: Banach lattices, modulared Banach spaces, Banach algebras, operator spaces...

In this talk, beside the category of Banach spaces we shall consider only that of Banach lattices.

If $(E_i) =$ Banach lattices, then $\ell_\infty(E_i; i \in I)$ is also a Banach lattice and $c_{0,\mathcal{U}}(E_i; i \in I)$ is a closed order ideal of this Banach lattice.
Thus the quotient $\prod_{\mathcal{U}} E_i$ has a Banach lattice structure too.

The inf operation on the ultraproduct $\prod_{\mathcal{U}} E_i$ is given by

$$[x_i]_{\mathcal{U}} \wedge [y_i]_{\mathcal{U}} = [x_i \wedge y_i]_{\mathcal{U}}$$

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Classes which are closed under ultraproducts

The following classes of normed structures are known to be closed under ultraproducts:

- $C(K)$ spaces (as Banach lattices or as Banach algebras)
- L_p-spaces, $1 \leq p < \infty$ (Banach lattices) [Krivine,Henson-Moore]
- Nakano spaces $L_{p(\cdot)}$ (or Lebesgue spaces with variable exponents). The class $\mathcal{N}_K$ of Nakano spaces with exponent function taking values in a given compact set $K \subset [1, \infty)$ is closed by ultraproducts. (as Banach lattices)
- Preduals of von Neumann algebras. [U. Groh]
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Classes which are not closed under ultraproducts, but...

The following classes of normed structures are not even closed under ultrapowers, but a suitable enlargement is closed under ultraproducts:

- $L_p + L_q$-spaces, $1 \leq p \neq q < \infty$; but the class of "generalized sums" $L_p(\Omega_1) + L_q(\Omega_2)$ (Ω_1, Ω_2 subsets of the same measure space) is closed under ultraproducts. [YR]
- Orlicz spaces. But the class of Musielak-Orlicz spaces (generalized Orlicz spaces with variable Orlicz function) satisyfing an prescribed uniform Δ_2-estimate is closed under ultraproducts. [Dacunha-Castelle]
- $L_p(L_q)$-spaces. But the class BL_pL_q of Banach lattices isomorphic to a band of some $L_p(L_q)$-space, $1 \leq p \neq q < \infty$ is closed under ultraproducts. [M. Levy, Y.R]

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The classical use of ultraproducts in BSG

Roughly speaking, using ultrapowers or ultraproducts allows to pass from local and approximate properties to global and exact ones. This is illustrated by the well known relationship to finite representability.

Definition

A normed vector space X is **finitely representable** into a normed vector space Y if for every finite dimensional subspace E of X and every $\epsilon > 0$ there is a linear map $T : E \to Y$ with

$$\forall x \in E, (1+\epsilon)^{-1}\|x\| \leq \|Tx\| \leq (1+\epsilon)\|x\|$$

(we say that T is an $(1+\epsilon)$-embedding of E into Y)

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Fact

Every ultrapower $X_{\mathcal{U}}$ of a normed space X is finitely representable in X.

Indeed given $E \subset X_{\mathcal{U}}$ a subspace with a basis $(e_1, e_2, \dots e_n)$
If $e_k = [e_{k,i}]_{\mathcal{U}}$, let $T_i : E \to X$ be the linear map such that $T_i e_k = e_{k,i}$.
Clearly $\|T_i x\|_X \xrightarrow[i,\mathcal{U}]{} \|x\|_E$ for each $x \in E$,
Assuming $\|e_{k,i}\| = \|e_k\| \ \forall k, i$ then

$$\sup_i \|T_i\| \leq K = \sup\{\sum |\lambda_k| \|e_k\| \mid \|\sum_k \lambda_k e_k\| = 1\}$$

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Fact

Every ultrapower $X_{\mathcal{U}}$ of a normed space X is finitely representable in X.

Indeed given $E \subset X_{\mathcal{U}}$ a subspace with a basis $(e_1, e_2, \dots e_n)$
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Proof: Sufficiency is clear by $X_{\mathcal{U}}$ f.r. X. Let's prove necessity:

For every $E \subset Y$ of dimension d_E let $T_E : E \to X$ be an $(1+\epsilon_E)$-linear embedding, with $\epsilon_E = 1/d_E$.

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The language of Banach spaces

In Logic a language is a set of formulas build in a precise way. For Banach space theory we define successively:

Atomic formulas:

" $\|\sum_{i=1}^{n} \lambda_i x_i\| \leq r$ " or " $\|\sum_{i=1}^{n} \lambda_i x_i\| \geq r$ "
where $x_1, \dots x_n$ are (vector) variables and $\lambda_1 \dots \lambda_n, r$ are (scalar) constants.

Basic formulas (quantifier free positive formulas)

Are build recursively from atomic formulas using the connective $\wedge$ ("AND") and $\vee$ ("OR"), but never $\neg$ ("NOT").

General formulas (prenex form)

Obtained by bounding certain variables with "bounded quantifiers":
$\forall_r x\ \phi(x, x_1, \dots x_n)$ is true if for all x with $\|x\| \leq r$, $\phi(x, x_1, \dots x_n)$ is true
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A logic interpretation of finite representability

A *sentence* is a formula which has no free variable. A sentence is *universal* if it contains only universal quantifiers (∀).

A *weakening* of a formula ϕ is a formula obtained by relaxing all the conditions appearing in ϕ:

$$\|\ldots\| \leq r \text{ and } \exists_r \text{ become } \|\ldots\| \leq r' \text{ and } \exists_{r'} \text{ with } r' > r$$
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We say that a sentence is approximately true in X if all its weakenings are true in X.

Fact

Let X, Y be normed spaces. Then the following assertions are equivalent:

- *Y is finitely representable in X*
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Ties between a normed space and its ultrapowers

It results from the precedings that a normed space and its ultrapowers satisfy the same universal sentences. However it is known since a long time that their relationship is far more intimate:

Theorem (Łoś; Henson)

Let X be a normed space and $X_{\mathcal{U}}$ an ultrapower of X.

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Elementary equivalence

Definition

Two normed spaces X and Y are said to be *elementarily equivalent* if they have linearly isometric ultrapowers $X_{\mathcal{U}}$ and $Y_{\mathcal{V}}$. The class of spaces Y that are elementary equivalent to X is called the *elementary class* of X.

Clearly if X and Y are elementarily equivalent they share the same set of approximately true sentences. The converse is true, and this is an hard theorem of Shelah (Henson for the normed spaces version):

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Let X, Y be normed spaces. The following assertions are equivalent:

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The ultraroot problem

If Z is linearly isometric to some ultrapower of X we call X an *ultraroot* of Z.

In this section we address the general loose question:

- given a class $\mathcal{C}$ of normed spaces, can we identify the class $\mathcal{C}_{ur}$ of normed spaces X which are ultraroot of some member of $\mathcal{C}$?

Note that if $\mathcal{C}$ is the class of all the ultrapowers of Y then $\mathcal{C}_{ur}$ is the elementary class of Y.

Ultraroots: the Banach space case

Within the list of classes of Banach spaces which are closed under ultraproduct, very few are classically known to be closed under ultraroots :

- L_p-spaces, $1 \leq p < \infty$ (Henson)
- L_1-preduals, and various subclasses (in particular, $C(K)$ spaces) (Heinrich, 1981)
- p-direct sums of spaces $L_p(H_i)$, $1 < p < \infty$, H_i Hilbert (Y.R., 2004)

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- L_1-preduals, and various subclasses (in particular, $C(K)$ spaces) (Heinrich, 1981)
- p-direct sums of spaces $L_p(H_i)$, $1 < p < \infty$, H_i Hilbert (Y.R., 2004)

Tools for showing closure under ultraroots

Tools for showing a class to be closed under ultraroots are quite scarce:

- the class is closed under subspaces
  Indeed a Banach space embeds canonically into any of its ultrapowers.
  Example: Hilbert spaces
- the class is closed under contractive projections (for a class of reflexive Banach spaces).
  Indeed a reflexive Banach spaces is 1-complemented in any of its ultrapowers.
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- the dual class is closed by ultraroots (class of reflexive Banach spaces).
  Indeed if $X_{\mathcal{U}}$ is a reflexive Banach space then $(X_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$.
- the class is a script-class
  Given a collection $\mathcal{C}_0$ of finite dimensional normed vector spaces, a normed vector space X is script-$\mathcal{C}_0$ iff for every finite dimensional subspace E of X and every $\epsilon > 0$, there exists a finite-dimensional subspace F containing E and $(1+\epsilon)$-isomorphic to a member of $\mathcal{C}_0$.
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Ultraroots: the Banach lattice case

- In the category of Banach lattices we have also a notion of ultraproduct.
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The following classes of Banach lattices are closed under ultraroots (in Banach lattice sense)

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A transfer theorem

Definition

A Banach lattice X has property DPIU iff every linear isometry from X into any of its ultrapowers preserves disjointness.

Examples

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Theorem

If a Banach space X has an ultrapower $X_{\mathcal{U}}$ linearly isometric to a Banach lattice Y with DPIU, then X itself is linearly isometric to a Banach lattice E such that $E_{\mathcal{U}}$ is lattice-isomorphic to Y.

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An example: the elementary class of $L_p(0,1)$

Analysis

- Since the class of L_p-spaces is closed under ultrapowers and ultraroots, the elementary class of $L_p(0,1)$ contains only L_p-spaces.
- Since the measure space $([0,1], \mathrm{Lebesgue})$ has no atom, the Banach lattice $L_p(0,1)$ has no atom as well. (An element e of a Banach lattice is an atom if $|x| \leq e \Longrightarrow \exists \rho, x = \rho e$).
- If X is a Banach lattice which does not contain ℓ_∞^n uniformly, then $X_{\mathcal{U}}$ has atoms iff X has.
- Consequently all ultrapowers of $L_p(0,1)$ are atomless and so are their ultraroots (in Banach lattice sense). But if $p \neq 2$ an ultraroot of a L_p space in Banach sense is linearly isometric to an ultraroot in Banach lattice sense. Hence all Banach spaces that are elementarily equivalent to $L_p(0,1)$ are L_p-spaces of an atomless measure space.

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Synthesis

- Now by general results in Logic, every infinite dimensional Banach space is elementary equivalent to a separable one.
- The only separable L_p space of an atomless measure space is $L_p(0,1)$ (up to an isometric isomorphism).
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Uncountable categoricity

It is known that the elementary class of any infinite dimensional Banach spaces has members of any infinite density character.

It is also known that if for some uncountable cardinal κ, the elementary class contains only one member (up to linear isometries) then the same is true for all uncountable cardinals (the class is said "uncountably categorical").

A trivial example is the elementary class of $\ell_2(\mathbb{N})$.

This is the class $\mathcal{H}$ of infinite dimensional Hilbert spaces:
[The proof goes like that for $L_p(0,1)$: the class $\mathcal{H}$ is closed under ultrapowers and ultraroots, and $\ell_2(\mathbb{N})$ is its unique separable member].

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Non Hilbertian examples

Recently the structure of Banach spaces with uncountably categorical elementary class has been studied by Shelah and Usviatsov, but with essentially no concrete example but the Hilbert spaces.

With C. Ward Henson we tried to build nontrivial examples. All these examples are separable spaces X such that all the ultrapowers of X and their ultraroots have the form

$$X \oplus H$$

where H is some Hilbert space (depending on the ultrapower or the ultraroot) while the norm on the direct sum does depends on the H-component only through its norm, i.e.

$$\|x \oplus h\| = |||x \oplus \|h\| \,|||$$

where $|||\cdot|||$ is a norm on $X \oplus \mathbb{R}$.
Note that for uncountable κ, $X \oplus H$ has density character κ iff H has, and there is only one Hilbert space of density character κ.

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Nakano type examples

N is a discrete Nakano space (variable exponent space) $\ell_{(p_n)}$ with

1. $p_n > 2$ and $p_n \to 2$
2. for no $c > 0$ the series $\sum_{n=1}^{\infty} c^{\frac{2p_n}{|p_n - 2|}}$ is convergent

note that condition (2) tells that the convergence $p_n \to 2$ is slow. By a result going back to Nakano (1951) it implies that N is not linearly isomorphic to an Hilbert space.

Lemma

Condition $p_n \to 2$ implies that the ultrapowers of N are of the form $N \oplus \mathcal{H}$, where $\mathcal{H}$ is an Hilbert space. Moreover the direct sum $N \oplus \mathcal{H}$ is modular, i. e. the norm is given by the convex modular

$$\Theta(x+h) = \sum_n |x_n|^{p_n} + \|h\|^2 = \Theta_N(x) + \Theta_H(h)$$

by Luxemburg formula $\|x+h\| = \inf\{c > 0 : \Theta(c^{-1}(x+h)) \leq 1\}$.

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$$\Theta(x+h) = \sum_n |x_n|^{p_n} + \|h\|^2 = \Theta_N(x) + \Theta_H(h)$$

by Luxemburg formula $\|x+h\| = \inf\{c > 0 : \Theta(c^{-1}(x+h)) \leq 1\}$.

Nakano type examples

N is a discrete Nakano space (variable exponent space) $\ell_{(p_n)}$ with

1. $p_n > 2$ and $p_n \to 2$
2. for no $c > 0$ the series $\sum_{n=1}^{\infty} c^{\frac{2p_n}{|p_n-2|}}$ is convergent

note that condition (2) tells that the convergence $p_n \to 2$ is slow. By a result going back to Nakano (1951) it implies that N is not linearly isomorphic to an Hilbert space.

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Corollary

Let $N = \ell_{(p_n)}$, $p_n \to 2$. The class of modular direct sums $N \oplus_m H$, H any Hilbert space, is closed under ultrapowers (and even by ultraproducts).

Indeed the modular direct sum "pass to the ultrapower" and is associative:

$$(N \oplus_m H)_{\mathcal{U}} = N_{\mathcal{U}} \oplus_m H_{\mathcal{U}} = (N \oplus_m \mathcal{H}) \oplus_m H_{\mathcal{U}} = N \oplus_m (\mathcal{H} \oplus_m H_{\mathcal{U}})$$

and $\mathcal{H} \oplus_m H_{\mathcal{U}} = \mathcal{H} \oplus_2 H_{\mathcal{U}}$ is an Hilbert space. □

We have now a structure result for embeddings from $N \oplus_m H$ into $N \oplus_m K$:

Lemma (Rigidity of embeddings)

Assume $\forall n, p_n > 2$. Let H, K be Hilbert spaces. Then every linear isometric embedding from $N \oplus_m H$ into $N \oplus_m K$ sends N onto N and H into K.

In particular every linear isometry from N into N is surjective.

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Proposition

Assume $2 < p_n \to 2$. The class of spaces $N \oplus_m H$, H Hilbert space, is closed under ultraroots.

Corollary

The elementary class of N consists only of direct sums $N \oplus_m H$, H Hilbert spaces and is thus uncountably categorical.

Remark

In fact, the elementary class of N consists exactly of all the direct sums $N \oplus_m H$, H Hilbert spaces.

Indeed, for any uncountable κ, both the elementary classes of N and $N \oplus H$ contain a member of density character κ, each of which is a direct sum $N \oplus_m H_\kappa$, H_κ Hilbert of density κ. But H_κ is unique up to a linear isometry.

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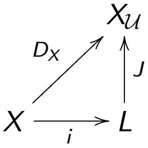
Proof of the proposition

Assume $J : L = N \oplus_m H \to X_{\mathcal{U}}$ is a linear isometry onto; set $i = J^{-1}D_X$, this gives a commutative diagramme,



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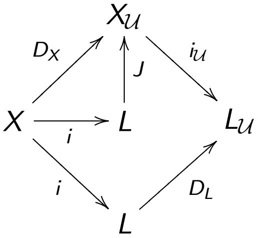

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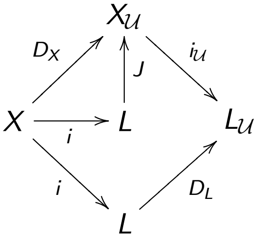

Clearly $i_{\mathcal{U}} D_X(X) = D_L i(X) \subset i_U(X_U) \cap D_L(L)$. It is an easy exercise to show that the last inclusion is an equality.

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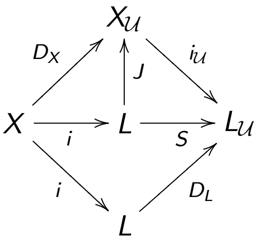
Assume $J : L = N \oplus_m H \to X_{\mathcal{U}}$ is a linear isometry onto; set $i = J^{-1} D_X$, this gives a commutative diagramme, that we insert into a bigger one. We complete by setting $S = i_{\mathcal{U}} J$.



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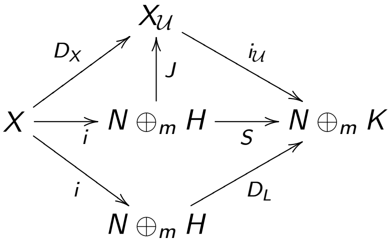

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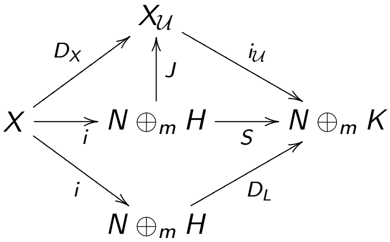

Recall that $i_{\mathcal{U}}(X_{\mathcal{U}}) \cap D_L(N \oplus_m H) = i_{\mathcal{U}} D_X(X) = D_L i(X)$
Since (embedding structure) $N = S(N) = i_{\mathcal{U}}(J(N))$, and $N = D_L(N)$ it follows that $N \subset i(X)$.

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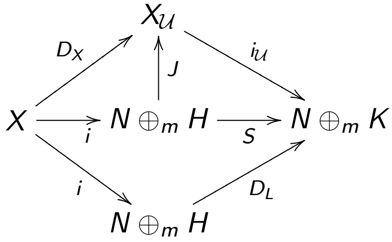

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Since (embedding structure) $N = S(N) = i_{\mathcal{U}}(J(N))$, and $N = D_L(N)$ it follows that $N \subset i(X)$.
Let $\pi^N : N \oplus_m H \to N$ the 1st coordinate projection and $\pi_0^N = \pi^N |_{i(X)}$.
$H_0 := \ker \pi_0^N \subset \ker \pi^N = H$ is an Hilbert sp. and $i(X) = N \oplus_m H_0$.

Thank you for your attention!