INTERPOLATION OF HARDY SPACES ON CIRCULAR DOMAINS

in collaboration with Radosław Szwedek

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Motivation

Theorem (Chalendar–Partington 2002)

Let $p \in (2, \infty)$ *and* $f \in H^p(\partial G)$. *Then*

$$H^p(\partial G) = \big(H^2(\partial G), H^\infty(\partial G)\big)_{\theta,p} \quad \frac{1}{p} = \frac{1-\theta}{2}.$$

I. Chalendar, J. R. Partington
Interpolation between Hardy spaces on circular domains with application to approximation
Arch. Math. 78 (2002), 223–232.

Hardy Spaces on a disc

Peter Jones' result

Theorem (Jones 1983)

If $p_0 \in (0, \infty)$*, then*

$$\left(H^{p_0}, H^{\infty}\right)_{\theta,q} = H^{p,q} \quad \frac{1}{p} = \frac{1-\theta}{p_0}$$

$$\left(H^{p_0}, H^{\infty}\right)_{\theta} = H^{p} \quad \frac{1}{p} = \frac{1-\theta}{p_0}.$$

P. Jones,

L^{∞} estimates for the $\bar{\partial}$-problem on a half plane

Acta Math. 150 (1983), 137–152.

Köthe function spaces

X is a **Köthe function spaces** if

- $X \subset L^0(\Omega, \Sigma, \mu)$ – the space of real valued measurable functions on Ω. The order $|x| \leq |y|$ means $|x(\omega)| \leq |y(\omega)|$ for μ-almost all $\omega \in \Omega$.
- There exists $u \in X$ with $u > 0$ μ-a.e. on Ω and $|x| \leq |y|$ with $x \in L^0(\Omega)$ and $y \in X$ implies $x \in X$ with $\|x\|_X \leq \|y\|_X$.
- If $x \in X$, then for any g equimeasurable with f, $\|f\|_X = \|g\|_X$

We will consider **complex** Köthe function spaces.

The role model for X are Lebesgue spaces, other important examples

- Orlicz spaces
- Lorentz spaces
- Marcinkiewicz spaces

Köthe function space X is **maximal** (or has the **Fatou property**) if whenever $\{x_n\}$ is a norm bounded sequence in X such that $0 \leq x_n \uparrow x \in L^0(\Omega)$, then $x \in X$ and $\|x\| = \lim_{n\to\infty} \|x_n\|$.

Hardy spaces $HX(\mathbb{D})$ and $HX(\mathbb{T})$

Let X be the Köthe space on $\mathbb{T} := [0, 2\pi)$. We define **Hardy spaces** $HX(\mathbb{D})$ and $HX(\mathbb{T})$ in the following way.

$$HX(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{hx(\mathbb{D})} := \sup_{r \in [0,1)} \|f_r\|_X < \infty \right\}$$

$$HX(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}, m) : \widehat{f}(n) = 0 \text{ if } n < 0 \text{ and } f^* \in X(\mathbb{T}) \right\}$$

where $f^*(t) := \lim_{r \to 1^-} f(re^{it})$, $f_r(t) := f(re^{it})$, $t \in \mathbb{T}$.

Theorem (Mastyło–M. 2009)

If X is maximal and separable then

$$HX(\mathbb{D}) = HX(\mathbb{T}).$$

M. Mastyło, P. Mleczko
Absolutely summing multipliers on abstract Hardy spaces
Acta Math. Sin. 25 (2009), no. 6, 883–902.

Interpolation of Hardy spaces on a disc

Theorem (Kisialov–Xu 1992–1999)

If (X_1, X_2) is a pair of maximal Köthe function spaces on $\mathbb{T}$, then for any interpolation functor $\mathcal{F}$ the following formula holds

$$\mathcal{F}(HX_1(\mathbb{T}), HX_2(\mathbb{T})) = H\mathcal{F}(X_1(\mathbb{T}), X_2(\mathbb{T})).$$

S. V. Kisliakov
Interpolation of H^p spaces: some recent developments
Function spaces, interpolation spaces, and related topics, Israel Math. Conf. Proc. 13 (1999), 102–140.

Q. Xu
Notes on interpolation of Hardy spaces
Ann. Inst. Fourier 42 (1992), 875–889.

Interpolation. Banach couple

We call $\vec{A} := (A_0, A_1)$ a **Banach couple** if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}.$$

For a given Banach couple $\vec{A}$, we define spaces

$A_0 \cap A_1$ with the norm $\|a\|_{A_0 \cap A_1} = \max\big\{\|a\|_{A_0}, \|a\|_{A_1}\big\}$ **(intersection)**

$A_0 + A_1$ with the norm $\|a\|_{A_0 + A_1} = \inf\limits_{a = a_0 + a_1} \big\{\|a_0\|_{A_0} + \|a_1\|_{A_1}\big\}$ **(sum)**

Interpolation. Functors

By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that

$$T|_{A_0} \in L(A_0, B_0) \text{ and } T|_{A_1} \in L(A_1, B_1)$$

Definition

By an **interpolation functor** we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$ such that

$$A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1 \text{ for any } \vec{A} \in \vec{\mathcal{B}}$$

$$T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B}) \text{ for any } \vec{A}, \vec{B} \in \vec{\mathcal{B}} \text{ and } T\colon \vec{A} \to \vec{B}.$$

Examples

The real method $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$.

The complex method $\mathcal{F}(\cdot) = (\cdot)_{\theta}$.

Hardy spaces on a circular domain

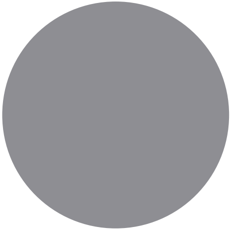
Circular domains G

A domain G is called a circlular domain if

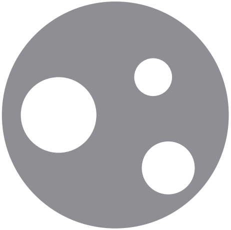
$$G = \mathbb{D} \setminus \bigcup_{i=1}^{n} (a_i + r_i\mathbb{D}),$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a_i's belong to $\mathbb{D}$ $(i = 1, 2, \ldots, n)$ and r_i's are numbers from the interval $(0, 1)$ such that $\mathbb{D}_n := a_i + r_i\mathbb{D} \subset \mathbb{D}$ for each $i \in \{1, 2, \ldots, n\}$. The role model for a circular domain is an anulus $\mathbb{A}$, that is $\mathbb{A} = \big\{z \in \mathbb{C} : |z| \in (r_0, 1),\ r_0 \in (0, 1)\big\}$.

disc



circular domain



Definition

Let G be a circular domain. A function $f \in H(G)$ belongs to $H^p(G)$ if there exists a harmonic majorant of $|f|^p$ on G.

Theorem (Rudin 1955)

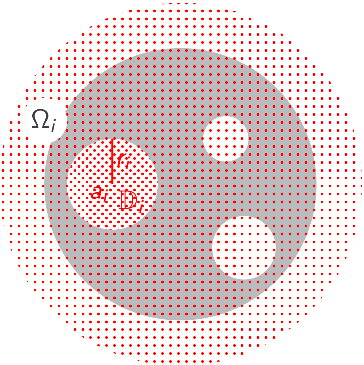
$$H^p(G) = H^p(\mathbb{D}) \oplus \bigoplus_{i=1}^{n} H_0^p(\Omega_i),$$

where $H_0^p(\Omega_i)$ *consists of functions* $f \in H^p(\Omega_i)$ *which vanish at infinity.*

W. Rudin
Analytic functions of class H^p
Trans. Amer. Math. Soc. 78 (1955), 46–66.

D. Sarason
The H^p spaces of an annulus
Mem. Amer. Math. Soc. No. 56 (1965) pp. 78.

Circular domains – throughout insight



$$G := \mathbb{D} \setminus \bigcup_{i=1}^{n} (a_i + r_i \mathbb{D}),$$

It can easily seen that if Ω_n are the complements of the closure of $\mathbb{D}_n$, then

$$G = \mathbb{D} \cap \bigcap_{i=1}^{n} \Omega_n.$$

Fundamental rôle play *Riemann functions* $\varphi_i : \mathbb{D}_i \to \overline{\mathbb{C}}$ given by

$$\varphi_i(z) = \frac{r_i}{z - a_i}, \quad z \in \mathbb{D}_i,\ i = 1, \ldots, n.$$

Hardy spaces $HX(G)$

Let G be a circular domain and X be a Köthe function space. For a domain Ω_i, the Hardy space $HX(\Omega_i)$ consists of functions $f \in H(\Omega_i)$ such that $f \circ \varphi_i^{-1} \in HX(\mathbb{D}_i)$ with the norm induced from $HX(\mathbb{D}_i)$, i.e.

$$\|f\|_{HX(\Omega_i)} = \|f \circ \varphi_i^{-1}\|_{HX(\mathbb{D})}, \quad f \in HX(\Omega_i).$$

The **Hardy space** $HX(G)$ is defined as follows

$$HX(G) := HX(\mathbb{D}) \oplus \bigoplus_{i=1}^{n} HX_0(\Omega_i),$$

with a norm given by

$$\|f\|_{HX(G)} = \max\{\|f_i\|_{HX(\Omega_i)} : f = f_0 + \cdots + f_n,\ i = 0, \ldots n\}.$$

Here HX_0 consists of functions from HX with a zero limit at infinity.

Hardy spaces $HX(\partial G)$

Let G be a circular domain and X be a Köthe function space. Let us also denote by $R(G)$ the set of rational functions with no poles in the closure of G.

The **Hardy space** $HX(\partial G)$ is the closure of $R(G)$ in the topology of $X(\partial G)$ with a norm given by

$$\|f\|_{HX(\partial G)} := \max\big\{\|f \circ \varphi_i^{-1}\|_{X(\mathbb{T})} : i = 0, 1, \dots, n\big\}, \quad f \in X(\mathbb{T}).$$

Note that we identify function from $R(G)$ with it's restriction to ∂G.

$HX(G) = HX(\partial G)$

Theorem (M.–Szwedek 2015)

Let X be a maximal and separable Banach lattice on $\mathbb{T}$. Then $R(G)$ forms a dense subset in $HX(G)$ and moreover

$$HX(G) = HX(\partial G).$$

$R(G)$ – the set of rational functions with no poles in the closure of G.

S. D. Fischer
Function theory on planar domains
John Wiley & Sons, New York 1983.

Theorem (M.–Szwedek 2015)

Let (X_0, X_1) be a couple of the Köthe function space on $\mathbb{T}$ and G be a circular domain. Then For any interpolation functor $\mathcal{F}$ the following formula holds

$$\mathcal{F}(HX_0(\partial G), HX_1(\partial G)) = H\mathcal{F}(X_0, X_1)(\partial G).$$

Applications

Corollary (M.–Szwedek 2015)

Let G be a circular domain and suppose that X is an interpolation space with respect to $\vec{A}$ and let $x \in HX(\partial G)$. If the inclusion map $\mathrm{id}\colon X \to M_\psi(\vec{A})$ *is bounded with a constant* $C := \|\mathrm{id}\colon X \to M_\psi(\vec{A})\|$, *then for all* $s, t, \varepsilon > 0$ *there exist* $x_0 \in HA_0$ *and* $x_1 \in HA_1$ *such that*

$$x = x_0 + x_1, \quad \|x_0\| \le C(1+\varepsilon)\tfrac{\psi(s,t)}{s}\|x\|_X, \quad \|x_1\| \le C(1+\varepsilon)\tfrac{\psi(s,t)}{t}\|x\|_X.$$

If $\psi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is non-decreasing in each variable and positively homogeneous (that is, $\psi(\lambda s, \lambda t) = \lambda\psi(s,t)$ for all $\lambda, s, t \ge 0$), then the *Marcinkiewicz space* $M_\psi(\vec{A})$ generated by $\vec{A}$ is defined

$$M_\psi(\vec{A}) := \left\{ a \in A_0 + A_1 : \|a\| := \sup_{s,t>0} \frac{K(s,t,a)}{\psi(s,t)} < \infty \right\}.$$

S. V. Kisliakov
Quantitative aspect of correction theorems
Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov 92 (1979), 182–191.

The proof

Main ingredients of the proof

Theorem (M.–Szwedek 2015)

If $\mathcal{F}$ is any bounded interpolation functor then

$$\mathcal{F}\Big(\bigoplus_{i}^{n} \vec{A}_i\Big) \simeq \bigoplus_{i=1}^{n} \mathcal{F}(\vec{A}_i)$$

for all Banach couples $\vec{A}_1, \ldots, \vec{A}_n$, where the corresponding constants may depend only on F and n.