On the mutually non isomorphic $\ell_p(\ell_q)$ spaces

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The begining

- Cembranos, P., Mendoza, J.: *Banach spaces of vector-valued functions,* Lecture Notes in Mathematics 1676. Springer-Verlag, Berlin(1997).

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- Motos J., Planells M. J., Talavera, C. F.: *On some iterated weighted spaces*, J. Math. Anal. Appl., **338**, 162–174(2008).
- Motos, J., Planells, M. J.: *On sequence space representations of Hörmander-Beurling spaces.* J. Math. Anal. Appl., **348**, no. 1, 395–403(2008).

- Triebel, H.: *Interpolation theory, function spaces, differential operators.* VEB Deutscher Verlag der Wissenschaften, Berlin (1978) and North-Holland Publishing Co., Amsterdam-New York(1978) (First editions), and Johann Ambrosius Barth, Heidelberg(1995) (Second edition).

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- Pełczyński: *Let* $1 \leq p_0, p_1 \leq \infty$ *and* $1 < q_0, q_1 < \infty$. *Then the spaces* $\ell_{p_0}(\ell_{q_0})$ *and* $\ell_{p_1}(\ell_{q_1})$ *are isomorphic if and only if* $p_0 = p_1$ *and* $q_0 = q_1$.

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- Question: What about the extreme cases for q_0, q_1 ?
- Question (Félix Cabello, 2009): *are the Banach spaces* $\ell_\infty(c_0)$ *and* $c_0(\ell_\infty)$ *isomorphic?*

Subspaces of $\ell_p(\ell_q)$ for $1 \leq p, q < \infty$

- **Question 1:** *For which* p_0, p_1, q_0, q_1*, with* $1 \leq p_0, p_1, q_0, q_1 < \infty$*, does one have*

$$\ell_{p_0}(\ell_{q_0}) \supset \ell_{p_1}(\ell_{q_1}) \ ?$$

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- **Answer:** Only in the trivial cases.

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- **Question (E.M. Galego):** Does one have

$$\ell_1(c_0) \supset c_0(\ell_1) \ ?$$

Complemented subspaces of $\ell_p(\ell_\infty)$

- **Theorem:** Let $1 \leq p, q < \infty$, then $\ell_p(\ell_\infty) \underset{(c)}{\supset} \ell_q$ if and only if $q = p$.

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- **Theorem:** Let $1 \leq p, q < \infty$, then $\ell_p(\ell_\infty) \underset{(c)}{\supset} \ell_q$ if and only if $q = p$.
- **Proposition:** For all q, $1 < q < \infty$, $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_2$.

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Complemented subspaces of $\ell_\infty(\ell_q)$

- **Proposition:** Let $1 \leq p, q \leq \infty$, then the following are equivalent:
  1. $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_p$.
  2. $(\sum_{n=1}^{\infty} \oplus \ell_q^n)_{c_0} \underset{(c)}{\supset} (\sum_{n=1}^{\infty} \oplus \ell_p^n)_{c_0}$.
  3. $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_\infty(\ell_p)$.

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- **Proposition:** Let $1 < p, q < \infty$. If $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_p$ then either $q \le p \le 2$ or $2 \le p \le q$.

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- **Proposition:** Let $1 < p, q < \infty$. If $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_p$ then either $q \leq p \leq 2$ or $2 \leq p \leq q$.
- **Theorem:** Let $1 \leq q_0, q_1 \leq \infty$. If $\ell_\infty(\ell_{q_0}) \approx \ell_\infty(\ell_{q_1})$ then $q_0 = q_1$

Open Problems

- **Proposition:** Let $1 \leq p < \infty$, then $\ell_\infty(\ell_1) \underset{(c)}{\supset} \ell_p$ if and only if $p = 1$.

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- **Problem 1:** Given $1 < q < \infty$, is it true that for $1 < p < \infty$ $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_p$ if and only if either $p \in [q, 2]$ or $p \in [2, q]$?

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- **Proposition:** Let $1 \leq p < \infty$, then $\ell_\infty(\ell_1) \underset{(c)}{\supset} \ell_p$ if and only if $p = 1$.
- **Problem 1:** Given $1 < q < \infty$, is it true that for $1 < p < \infty$ $\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_p$ if and only if either $p \in [q, 2]$ or $p \in [2, q]$?
- **Problem 2:** *For which q_0, p_1, q_1, with $1 \leq q_0 < \infty$ and $1 \leq p_1, q_1 \leq \infty$, does one have*

$$\ell_\infty(\ell_{q_0}) \underset{(c)}{\supset} \ell_{p_1}(\ell_{q_1})?$$

In particular, for which q, r, with $1 < q, r < \infty$, does one have

$$\ell_\infty(\ell_q) \underset{(c)}{\supset} \ell_r?$$

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Thank you very much!