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About Mann type methods

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The Contraction Principle says not only that a self-contraction defined on a complete metric space has a unique fixed point, but also that any orbit converges to this point. This fact is not true, in general, for nonexpansive mappings.

One of most celebrated fixed point Theorem for nonexpansive mappings [i.e. mappings T for which d(Tx,Ty)≤ d(x,y)] is of course that of Kirk

Theorem (W. A. Kirk Amer. Math. Monthly 72 (1965) 1004-1006) Let E be a reflexive Banach space and let K be a closed, convex, bounded subset of E having normal structure.
Then any nonexpansive self-mapping defined on K has at least a fixed point.

However, easy examples show that the sequence of successive approximations

$$x_{n+1} = Tx_n$$

can be not convergent, also if T is a self-mapping defined on a closed convex and bounded subset of a Hilbert space.

Nevertheless, as shown by **Krasnoselskii (Uspehi Mat. Nauk (N.S.) 10 (1955) 123-127,** we can obtain a convergent sequence of successive approximations if instead of T we consider the averaged mapping

$$T_{1/2} = \frac{1}{2}(I + T),$$

The result found by Krasnoselkii says that

If the nonexpansive mapping T is defined on a closed, convex and bounded subset of a uniformly convex Banach space and has compact range, then the sequence

$$x_{n+1} = T_{1/2}x_n = \frac{1}{2}(x_n + Tx_n),$$

is strongly convergent.

More in general, if K is a closed convex subset of a normed linear space X, a generalization of the above method is due to Schaefer:

Theorem (H. Schaefer, Jber. Deutsch. Math. Verein. 59 (1957) 131-140) ***Let K be a closed, convex and bounded subset of a Banach space X***

and let T be a nonexpansive self-mapping defined on K. Let α be fixed in [0,1]. Consider

$$T_\alpha = \alpha T + (1-\alpha)I$$

and

$$x_{n+1} = T_\alpha^n x_o = (1-\alpha)x_n + \alpha T x_n$$

the sequence of successive approximations for T_α. Then

- *(i) If X is strictly convex, Fix(T) is convex.*
- *(ii) If X is uniformly convex and T(K) is compact, the orbit x_n converges strongly to a fixed point of T.*
- *(iii) If X is a real Hilbert space and T is weakly continuous on K, x_n converges weakly to a fixed point (depending on α and x_o)*

REMARKS:

- **The result (iii) was improved by Opial, Bull. Amer. Math. Soc. 73 (1967) 591-597:**

Let T (with Fix(T) non empty) a nonexpansive self mapping defined on a closed, convex subset K of an uniformly Banach space with duality mapping weakly continuous. Then for each x_o in K and for each α in [0, 1], the orbit

$$x_{n+1} = T_\alpha^n x_o = (1-\alpha)x_n + \alpha T x_n$$

weakly converges to a fixed point of T.

[And the strong convergence? In general no. The beautiful example of Genel and Lindestrauss [Israel J. Math. 22 (1975) 81-86] exhibits a nonexpansive mapping T on the unitary ball of the Hilbert space l^2 for which the Krasnoselskii's sequence of the successive approximations

$$x_{n+1} = T_\alpha^n x_o = (1-\alpha)x_n + \alpha T x_n$$

c**onverges weakly but not strongly]**

- **The result (ii) of Schaefer's Theorem is the extension of the Krasnoselskii's Theorem from $T_{1/2}$ to T_α. Such result was extended to strictly convex spaces by Edelstein [Amer. Math. Montly 13 (1966) 507-510]**

At this point, the question to see if also the strict convexity could be removed, remained open for ten years. In 1976 this question was in the affirmative answered by Ishikawa.

It is in such context that the basic results about this theme have developed, almost simultaneously, sometimes overlapping.

So far we have mentioned the Krasnoselskii Algorithm

$$\boldsymbol{x_{n+1} = \frac{1}{2}(x_n + Tx_n)}$$

and the Schaefer algorithm

$$\boldsymbol{x_{n+1} = T_\alpha^n x_o = (1-\alpha)x_n + \alpha T x_n}$$

More or less in the same years is introduced the Mann-Dotson Algorithm, that generalizes the two previous, on which we want to dwell.

The process

$$\boldsymbol{x_{n+1} = (1-t_n)x_n + t_n T x_n \quad (*)}$$

is well known as Mann iterative process.

In the original paper of **Mann,** [1]“**Mean value methods in iteration” published on Proc. AMS, 4 (1953) 506-510,** (*) does not appear. The process studied by Mann is most more general than (*). It introduced an infinite triangular matrix A,

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & \ldots \\ a_{21} & a_{22} & & 0 & \ldots \\ \vdots & & \ddots & \vdots & \\ a_{n1} & a_{n2} & \cdots & 0 & \ldots \\ \ldots & \ldots & & \ldots & \ldots \end{bmatrix}$$

whose elements satisfy the following restrictions:

(A$_1$) $\boldsymbol{a_{ij} \geq 0 \quad \forall i, j}$

(A$_2$) $\boldsymbol{a_{ij} = 0 \quad \forall j > i}$

(A$_3$) $\boldsymbol{\sum_j a_{ij} = 1 \quad \forall i}$

(A$_4$) $\boldsymbol{\lim_n a_{nj} = 0 \quad \forall j}$

Starting with an arbitrary element x_1 one can then define the process

$$\begin{cases} v_n = \sum_{k=1}^{n} a_{nk} x_k \\ x_{n+1=T(v_n)} \end{cases}$$

This process is determinated by the initial point x_1, the matrix A and the mapping T. It can be denoted briefly by M(x_1, A, T) and can be regarded as a generalized iteration process because when A is the identity matrix I, the process M(x_1, I, T) is just the ordinary Banach-Caccioppoli-Picard's iteration x_{n+1}=T(x_n).

Mann proved that in case E is a Banach space and the domain of a continuous self –mapping T is a closed convex subset C of E, then the convergence of either x_n or v_n to a point y implies the convergence of the other to y and also implies Ty=y.

As a particular case of the general process M(x_1, A, T), Mann considered the Cesàro matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 1/2 & 1/2 & 0 & 0 & \dots \\ 1/3 & 1/3 & 1/3 & 0 & \dots \\ 1/4 & 1/4 & 1/4 & 1/4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

In this particular case the v_n of the general process

$$\begin{cases} v_n = \sum_{k=1}^{n} a_{nk} x_k \\ x_{n+1=T(v_n)} \end{cases}$$

becomes

$$v_{n+1} = \left(1 - \frac{1}{n+1}\right) v_n + \frac{1}{n+1} T v_n.$$

After 13 years, was published the paper of **W. G. Dotson jr.** [2]**"On the Mann iterative process", Trans. AMS 149 (1970) 65-73.**

It is in this paper that Dotson, probably inspired by the example of the Cesaro matrix, defined a

normal Mann process as a Mann process M(x_1, A, T) for which the matrix A satisfies not only

(A_1) $\boldsymbol{a_{ij} \geq 0 \quad \forall i, j}$

(A_2) $\boldsymbol{a_{ij} = 0 \quad \forall j > i}$

(A_3) $\boldsymbol{\sum_j a_{ij} = 1 \quad \forall i}$

(A_4) $\boldsymbol{\lim_n a_{nj} = 0 \quad \forall j}$

but also

(A_5) $\boldsymbol{a_{n+1,j} = \left(1 - a_{n+1,n+1}\right) a_{n,j} \quad \forall j = 1, \dots, n}$

(A6) either $\boldsymbol{a_{nn} = 1 \, \forall n}$ **or** $\boldsymbol{a_{nn} < 1 \, \forall n}$.

And even in such paper Dotson shown, besides other interesting results, that the sequence v_n in a normal Mann process M(x_1, A, T) satisfies

$$\boldsymbol{v_{n+1} = (1 - t_n) v_n + t_n T v_n}$$

where $t_n = a_{n+1,n+1}$.

It is well known that ***if T is a nonexpansive self mapping on a closed convex subset of a uniformly convex Banach space with a Frechèt differentiable norm and if Fix(T) is nonempty, then the sequence x_n generated by the Mann-Dotson algorithm***

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n$$

converges weakly to a point of Fix(T) if

$$\sum t_n(1 - t_n) = \infty$$

(S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces" JMAA 67 (1979) 274-276).

This convergence is in general not strong (there is the celebrated counter-example of **Genel and Lindestrauss, Israel J. Math. 22 (1975) 81-86).**

So, we are interesting all the results in which one can obtain strong convergence, of course either in some particular space or for some particular nonexpansive mapping.

Some of this results are the following:

Theorem (H. F. Senter and W. G. Dotson jr, Proc. AMS 44 (1974) 375-380) ***Suppose X is a***

uniformly convex Banach space, C is a closed, bounded, convex, nonempty subset of X and T is a nonexpansive self-mapping on C that satisfies the following
Condition (I): there is a nondecreasing function $f: [0,\infty) \to [0,\infty)$ *with* $f(0) = 0, f(r) > 0\ \forall r > 0, f(d(x, Fix(T))) \leq d(x, T(x))\ \forall x \in C$. *Then for any* x_1 *in C, the normal Mann-Dotson process*

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n$$

with $0 < a \leq t_n \leq b < 1$, *converges strongly to a point of Fix(T).*

With completely different techniques Ishikawa was able to prove the same result in any Banach space, not only in uniformly convex Banach spaces:

Theorem (S. Ishikawa, Proc. AMS 19 (1976) 65-71) *Suppose X is a Banach space, C is a closed, convex, nonempty subset of X and T is a nonexpansive self-mapping on C that satisfies either*
(i) T(C) is compact

or
(ii) Fix(T) is nonempty and T satisfies Condition (I).
Then for any x_1 in C, the normal Mann- Dotson process

$$\boldsymbol{x_{n+1} = (1 - t_n)x_n + t_n T x_n}$$

with $\boldsymbol{0 \leq t_n \leq b < 1, \sum t_n = \infty}$, *converges strongly to a point of Fix(T).*
(Very beautiful proof!)

Another nice result is due to Chidume in the Lebesgue spaces:

Theorem (C. E. Chidume, Proc. AMS 99 (1987) 283-288) ***Suppose X=L_p (or l_p), p greater or equal than 2, and C is a nonempty closed convex bounded subset of X. Suppose T is a self-mapping on C lipschitzian and strictly pseudo-contractive, i.e., there exists k>1 such that it results***

$$\boldsymbol{\|x - y\| \leq \|(1 + r)(x - y) - rk(Tx - Ty)\|}$$

for all positive r (some author call this mappings strong pseudocontractive)
Let t_n be a real sequence in]0,1[such that

(i) $\sum t_n = \infty$, $(ii) \sum t_n^2 < \infty$.

Then the iteration Mann-Dotson process converges strongly to the unique fixed point of T.

Remark 1: *The same* ***Chidume, together with Mutangadura (Proc. AMS 129 (2001) 2359-2363)*** *given the very nice example of a Lipschitz pseudo-contraction in the real plane with a unique fixed point for which every nontrivial Mann-Dotson sequence fails to converge.*
Underline explicitly that while the Mann-Dotson sequence does not converge to the fixed point of T in this example, the Ishikawa sequence does (this follows by a result of ***Liu Qihou, JMAA 148 (1990) 55-62)***

Another possible way to obtain strong convergence is to modify the process

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n$$

Recall here only few of such modifications:

- **Halpern process (B. Halpern Bull. AMS 73 (1967) 957-961)**

$$x_{n+1} = (1 - t_n)u + t_n T x_n$$

in which the convex combination used to define x_{n+1} is “anchored” to u. (In my opinion, the proof that the Halpern process for a nonexpansive self-mapping defined on the unit closed ball in a Hilbert space converges to the fixed point of minimum norm is simple but it is a jewel of the human mind

- **Ishikawa process (S. Ishikawa, Proc. AMS 44 (1974) 147-150)**

$$y_n = (1 - s_n)x_n + s_n T x_n$$
$$x_{n+1} = (1 - t_n)x_n + t_n T y_n$$

in which there is a double convex combination.

- **Moudafi process (A. Moudafi, JMAA 241 (2000) 46-55)**

$$\boldsymbol{x_{n+1} = (1 - t_n)f(x_n) + t_n T x_n}$$

where f is a contraction, called ***viscosity,*** since it represents a "brake" in the originary Mann-Dotson process.

The introduction of the viscosity term is not a simple formal object, but its significance is substantial. Indeed, Moudafi proved

Theorem. ***In the setting of Hilbert spaces, the sequence generated by the process***

$$\boldsymbol{x_{n+1} = (1 - t_n)f(x_n) + t_n T x_n}$$

strongly converges to the unique solution of the variational inequality

Find p in Fix(T) such that

$$< (I - f)p, x - p > \geq 0 \quad \forall x \in Fix(T),$$

in other words, the unique fixed point of the operator $P_{Fix(T)} \circ f$.

Remark: *A further modification of the Moudafi process is in the paper of my self and Hong Kun Xu* ***[23] G. M. and H. K. Xu, JMAA 318 (2006) 43-52. We introduced in the viscosity process a strongly positive linear bounded operator. This introduction is still not formal but gives another bridge between the metric fixed points theory and the calculus of variation. Indeed, we proved***

Proposition. ***Let H be a real Hilbert space. Consider on H a nonexpansive mapping T with a fixed point, a contraction f with coefficient α and a strongly positive linear bounded operator A with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \overline{\gamma}/\alpha$. Then, for suitable choice of the coefficients, the sequence generated by the iterative method***

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n$$

converges strongly to a fixed point p in Fix(T) which solves the variational inequality

$$< (A-\gamma f)p, x-p > \geq 0 \quad \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2} < Ax, x > -h(x)$$

where h is a potential function for γf.

- **CQ-method (K. Nakajo and W. Takahashi, JMAA 279 (2003) 372-379)**

In a Hilbert space,

$$\begin{cases} x_o \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ C_n = \{z \in C: \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C: < x_n - z, x_o - x_n > \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_o \end{cases}$$

Note that we call such process a CQ method for the Mann iteration process because at each step the Mann iterate y_n is used to construct the sets C_n and Q_n which are in turn used to construct the next iterate x_{n+1} and hence the name.

- **Shrinking projection method (W. Takahashi, Y. Takeuchi, R. Kubota, JMAA 341 (2008) 276-286)**

In a Hilbert space,

$$\begin{cases} x_o \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{C_{n+1}} x_o \end{cases}$$

The authors proved that if the sequence $\{\alpha_n\}$ is bounded above away from one, then the sequence $\{x_n\}$ generated by the shrinking projection method, converges strongly to $P_{Fix(T)}x_o$

Let's see how one can formulate the algorithm in Banach spaces.

Let X be a Banach space and X^* its dual space. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$\mathrm{J}\mathrm{x} = \{\mathrm{j} \in \mathrm{X}^*: < \mathrm{x}, \mathrm{j} > = \|\mathrm{x}\|^2 = \|\mathrm{j}\|^2\},$$

As well known, if C is a non-empty, closed and convex subset of a Hilbert space H, the metric projection P_C is nonexpansive. This fact characterizes Hilbert spaces and consequently it is not available in other Banach spaces.
In this setting, Alber and and Reich in 1994 introduced a generalized projection operator which is analogous to the metric projection in Hilbert spaces.
Suppose X a smooth Banach space. The generalized projection

$$\Pi_C: X \to C$$

is a map that assigns to an arbitrary point x

in X the set of minimal points with respect to the Liapunov functional
$\phi(x, y) = \|x\|^2 - 2 < x, Jy > +\|y\|^2$,
that is
$$\Pi_C(x) = \left\{\tilde{x} \in C: \phi(\tilde{x}, x) = \min_{y \in C} \phi(y, x)\right\}$$
If X is strictly convex and C ia a non-empty, convex and closed subset of X, then $\Pi_C(x)$ consists of at most one element for any x in X.
The basic properties of the generalized projection have been presented by Alber in the case of uniformly convex and uniformly smooth Banach spaces. Kamimura and Takahashi (SIAM J. Optim. 13 (2003)) proved them in reflexive, strictly convex and smooth Banach spaces. Li (JMAA 306 (2005)) studied the mapping Π_C in the wider setting of reflexive Banach spaces.
The Liapunov functional $\phi(x, y)$ is a particular case of the functionals used by Butnariu, Reich and Zalavski (J. Appl. Anal. 7 (2001)) to define the relatively non-expansive mappings in Banach spaces. Following them, we say that a mapping $T: C \to C$ is relatively non-expansive with respect to the convex function

$$h: X \to \mathbb{R}$$

if:

(1) C is a non-empty, convex and closed subset of a smooth Banach space X;

(2) the function h is lower semi-continuous on C;

(3) there exists a point c in C such that, for any x in C we have

$$D_h(c, Tx) \leq D_h(x, c),$$

where D_h stands for the Bregman distance given by

$$D_h(x, y) = h(y) - h(x) + h(x, x - y)$$

where $h(x, d)$ denotes the right-hand derivative of h at x in the direction d.

Note that under these assumptions, c is a fixed point of T (Butnariu,Reich, Zaslavski, NFAO 2003). We remark also that

if $h(x) = \|x\|^2$, then $D_{\|\cdot\|^2}(x, y) = \phi(x, y)$

Of course if X is a Hilbert space then

$$D_{\|\cdot\|^2}(x, y) = \phi(x, y) = \|x - y\|^2$$

The problem described above occurs in mathematics in various forms. Let us present some examples:

(a) If the mapping T is such that, for some c in C, $\|c - Tx\| \le \|c - x\|$ for all x in C, (i.e. mappings admitting a center, studied by Garcia Falset, Llorens-Fuster and Prus, NA 2007)), then it is relatively non-expansive with respect to the function $h(x) = \|x - c\|^2$.

(b) Each map T quasi-non-expansive with at least a fixed point (i.e. $\|Tx - p\| \le \|x - p\|$ for any x in C and p in Fix(T) is relatively nonexpansive with respect to any of the functions $h(x) = \|x - p\|^2$.

(c) Of course also each map T nonexpansive with a fixed point is relatively nonexpansive with respect to any of the functions $h(x) = \|x - p\|^2$.

But, in general, relatively nonexpansive mappings with respect to arbitrary convex functions h may not be mapping admitting a center (and so, a fortiori, quasi-nonexpansive).

Motivated by the ideas above, Matsushita and Takahashi (FPTA 2004) introduced the concept of ***relatively non-expansive*** mapping.

More precisely, let C be a non-empty closed convex subset of a smooth, strictly convex and reflexive Banach space X. Let T be a mapping from C into itself.

A point z in C is said to be an *asymptotic fixed point* of T if there exists a sequence $\{z_n\}$ in C converging weakly to z and $\lim_n \|z_n - Tz_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\tilde{F}(T)$. The mapping T is said to be a ***relatively nonexpansive*** mapping if the following three conditions are satisfied:

(R1) Fix(T) is nonempty;

(R2) $\varphi(u, Tx) \le \varphi(u, x)\ \forall u \in Fix(T),\ x \in C$

(R3) $\tilde{F}(T) = Fix(T)$.

Of course the concept of relatively non-expansive mapping is much stronger than the concept of relatively nonexpansive mapping with respect to the function $h(x) = \|x\|^2$. Nevertheless it seems yet interesting, due to the several examples of such a type of mapping given by Kohsaka and Takahashi (FPTA 2007).

By using the generalized projection Π_C several algorithms to find a fixed point of relatively non-expansive mappings has been developed, modifying the Mann, Ishikawa and hybrid algorithms. For the sake of completeness we quote here one of the latest results that covers many previous results:

Theorem (X. Qin, Y. Su, NA 2007). ***Let E be a uniformly convex and uniformly smooth Banach space, let C be a non-empty closed convex subset***

of E, let $T: C \to C$ ***be a relatively non-expansive mapping such that*** $Fix(T) \neq \emptyset$***. Assume that*** $\{\alpha_n\}$ ***and*** $\{\beta_n\}$ ***are sequences in [0,1]***

such that $\limsup_{n\to\infty} \alpha_n < 1$ ***and*** $\beta_n \to 1$***. Define a sequence*** $\{x_n\}$ ***in C by the following algorithm:***

$$\begin{cases} x_o \in C \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n J T x_n) \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n J T z_n) \\ C_n = \{v \in C: \varphi(v, y_n) \leq \ \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, z_n\} \\ Q_n = \{v\ C: < x_n - v, Jx_o - Jx_n \geq 0\} \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_o \end{cases}$$

Then, if T is uniformly continuous, we have that $\{x_n\}$ ***converges to*** $\Pi_{Fix(T)} x_o$.

With the aim to obtain simpler hybrid algorithm, valid in more general Banach spaces, we propose the following approach, by using the ideas above. First of all, note that if C is a non-empty, closed and convex subset of a reflexive Banach space X and if $f: C \times C \to \mathbb{R}$ is a functional such that for any $x_0 \in C$, $f(x_o,\cdot): C \to \mathbb{R}$
is strictly convex, weakly lower semi-continuous and $f(x_o, x_n)$ is not bounded if $\|x_n\| \to \infty$, then for any non-empty closed convex subset D of C, there exists exactly one element d in D such that $f(x_o, d) = inf_{x \in D} f(x_o, x)$.
We put $d = P_{f,D}(x_o)$ and, following Censor and Lent (JOTA, 1981), call $P_{f,D}$ ***the Bregman***

projection on D induced by f. Note that if $f(x,y) = \varphi(y,x)$ then $P_{f,D} = \Pi_D$, while if X is a Hilbert space and $f(x,y) = \varphi(x,y) = \|x - y\|^2$, then $P_{f,D} = \Pi_D = P_D$, where P_D is the classical metric projection.

Moreover, it easy to see that in a smooth, strictly convex Banach space X the functionals $f(x,y) = \varphi(x,y)$ and $f(x,y) = \|x - y\|^2$ restricted to $C \times C$, where C is a non-empty, closed and convex subset of X, have the following properties:

(i) for any $x_o \in C, f(x_o,\cdot): C \to \mathbb{R}$ is strictly convex, weakly lower semi-continuous and $f(x_o, x_n)$ $\|x_n\| \to \infty$ is not bounded if $\|x_n\| \to \infty$

(ii) If $d_n, x_o, d_o \in C$ are such that $d_n \to d_o$ weakly and $f(x_o, d_n) \to f(x_o, d_o)$, then $\|d_n\| \to \|d_o\|$.

Now we are able to give our results **(G. Lewicki and G.M.** ***On some algorithms in Banach spaces finding fixed points of nonlinear mappings*****,** NA 71 (2009) 3964-3972).

Theorem 1.

Let X be a reflexive, strictly convex, smooth Banach space with the property (K) of $\boldsymbol{Kade\check{c}-Klee}$***. Assume C is a non-empty, closed and convex subset of X.. Let*** $T: C \to C$ ***be a continuous relatively nonexpansive mapping with respect to*** $h(x) = \|x\|^2$***, i.e.*** $\exists c \in C$: $\varphi(c, Tx) \leq \varphi(c, x)\ \forall x \in C$.

Let $f: C \times C \to R$ ***be a functional such that for any*** $x_o \in C, f(x_o, \cdot): C \to R$ ***is strictly convex, weakly lower semi-continuous and*** $f(x_o, x_n)$ ***is not bounded if*** $\|x_n\| \to \infty$.

Assume furthermore that if $d_n, x_o, d_o \in C$ ***are such that*** $d_n \to d_o$ ***weakly and*** $f(x_o, d_n) \to f(x_o, d_o)$, ***then*** $\|d_n\| \to \|d_o\|$.

Let $\{a_n\} \subset [0,1]$ ***be a sequence or real numbers such that*** $b_o = limsup_n a_n < 1$.

For $n \geq 1$ ***define a sequence*** $\{x_n\} \subset C$ ***by the following algorithm:***

$$\begin{cases} x_o \in C, C_o = C \\ y_{n-1} = J^{-1}(a_{n-1}Jx_{n-1} + (1 - a_{n-1})JTx_{n-1}) \\ C_n = \{z \in C_{n-1}: \varphi(z, y_{n-1}) \leq \varphi(z, x_{n-1})\} \\ x_n = P_{f,C_{n-1}}x_{o.} \end{cases}$$

Then $\{x_n\}$ ***strongly converges to*** $d_o = P_{f,\cap_n C_n}x_o \in Fix(T)$

Remark 1.3. (a) In our Theorem the hypotheses on X are weaker than usual assumptions of uniform convexity and uniform smoothness.
For example, any strictly convex, reflexive and smooth Musielak_Orlicz space satisfies our assumptions (Hudzik, Kowalewski, Lewicki, Z. Anal. Anwendungen 2006)
while, in general, these spaces need not to be uniformly convex or uniformly smooth.
(b) The hypotheses on the mapping T are considerably weakened and they are satisfied also in the case of a continuous quasi-nonexpansive mapping T that 0 belongs to Fix(T)

(If T is a continuous quasi-nonexpansive mapping such that 0 does not belong to Fix(T) but $c \in Fix(T)$, the algorithm yet works with φ_c instead of φ, with the the same proof).

(c) The algorithm is simpler than the CQ-algorithm (since works only with the convex sets C_n, that are the intersection of C_{n-1} with a convex set and not with the operator Q_n

(d) The sequence $\{x_n\}$ depends on the function f and so the algorithm is extremely flexible.

(e) The Bregman projection $P_{f,D}$ induced by specific functions f are studied because of their intrinsic interest in various
applications (Alber and Butnariu, JOTA 1997)

Now we present a modification of the algorithm in Theorem 1.2 for a class of mappings wider than the class of the
relatively non-expansive mapping with respect to $h(x) = \|x\|^2$.

Theorem 2. ***Let X be a reflexive, strictly convex, smooth Banach space with the property (K). Assume C is a***

non-empty, closed and convex subset of X. Let $T: C \to C$ be a continuous mapping such that $\exists c \in C$: $\varphi(c, Tx) \le \varphi(c, x) + k\varphi(x, Tx)$ $\forall x \in C$.

Let $f: C \times C \to R$ be a functional such that for any $x_o \in C, f(x_o, \cdot): C \to R$ is strictly convex, weakly lower semi-continuous and $f(x_o, d_n) \to f(x_o, d_o)$, then $\|d_n\| \to \|d_o\|$.

Let $\{a_n\} \subset [0,1]$ be a sequence or real numbers such that $b_o = lim\, a_n = 0$.

For $n \ge 1$ define a sequence $\{x_n\} \subset C$ by the following algorithm:

$$\begin{cases} x_o \in C, \quad C_o = C \\ y_{n-1} = J^{-1}(a_{n-1}Jx_{n-1} + (1 - a_{n-1})JTx_{n-1}) \\ C_n = \{z \in C_{n-1}: \varphi(z, y_{n-1}) \le \varphi(z, x_{n-1}) + k\varphi(x_{n-1}, Tx_{n-1}\} \\ x_n = P_{f, C_{n-1}} x_{o.} \end{cases}$$

Then if f satisfies the assumptions of Theorem 1, $\{x_n\}$ strongly converges to

$$d_o = P_{f, \cap_n C_n} x_o \in Fix(T)$$

At least, we present a version of the previous Theorem in the case of Hilbert spaces in which we will not assume that $lim\, a_n = 0$.

Theorem 3. ***Let H be a Hilbert space. Assume C is a non-empty, closed and convex subset of H. Let $T: C \to C$ be a continuous mapping such that $\exists c \in C$ and k in [0,1) such that***

$$\|c - Tx\|^2 \le \|c - x\|^2 + k\|x - Tx\|^2 \quad \forall x \in C.$$

Let $\{a_n\} \subset [0,1)$.

For $n \ge 0$ define a sequence $\{x_n\} \subset C$ by the following algorithm:

$$\begin{cases} x_o \in C, \quad C_o = C \\ y_n = x_n + (1 - a_n)Tx_n \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \le \|z - x_n\|^2 + k\|x_n - Tx_n\|^2\} \\ x_{n+1} = P_{f,C_n} x_{o.} \end{cases}$$

Then $\{x_n\}$ strongly converges to

$$d_o = P_{f,\cap_n C_n} x_o \in Fix(T)$$

Remark. In Hilbert spaces any continuous quasi-strict pseudo-contraction (and thus, a fortiori, any strict pseudo-contraction)
satisfies the assumption on T in the Theorem.

Reasuming, we have seen some modified Mann's methods to get strong convergence

- **Halpern's method**

$$x_{n+1} = (1 - t_n)u + t_n T x_n$$

- **Ishikawa's method**

$$y_n = (1 - s_n)x_n + s_n T x_n$$
$$x_{n+1} = (1 - t_n)x_n + t_n T y_n$$

- **Moudafi's method**

$$x_{n+1} = (1 - t_n)f(x_n) + t_n T x_n$$

- **G.M. and Xu's method**

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T x_n$$

- **CQ-method**

In a Hilbert space,

$$\begin{cases} x_o \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T x_n \\ C_n = \{z \in C: \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C: \langle x_n - z, x_o - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_o \end{cases}$$

- **Shrinking projection method**

In a Hilbert space,

$$\begin{cases} x_o \in C \text{ chosen arbitrarily} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\} \\ x_{n+1} = P_{C_{n+1}} x_o \end{cases}$$

Lastly, we present a method that is *almost* the Mann's method $x_{n+1} = (1-t_n)x_n + t_n T x_n$, namely

$$x_{n+1} = (1-t_n)x_n + t_n T x_n - t_n s_n x_n$$

Under mild assumptions on the coefficients that permit s_n to be small what you want (but not zero), this little perturbation of the Mann's method, ensures the strong convergence to a point of Fix(T) of minimum norm (**G. M.and L. Muglia, FPTA 2015)**

Dziękuję bardzo za uwagę