Complex interpolation of Orlicz spaces with respect to a vector measure

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Joint work with R. del Campo, A. Fernández, F. Mayoral and F. Naranjo
(from Universidad de Sevilla)

Function Spaces XI
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• Let (Ω, Σ) be a measurable space and μ a σ-finite measure on (Ω, Σ).
If $1 \leq p_0 \neq p_1 \leq \infty$, $0 < \theta < 1$, $0 < q \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

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THEOREM (Beauzamy, Lecture Notes in Math. (1978))

Let $0 < \theta < 1$ and $1 < q < \infty$.

$$(A_0, A_1)_{\theta,q} \text{ is reflexive } \Leftrightarrow I : A_0 \cap A_1 \longrightarrow A_0 + A_1 \text{ is weakly compact.}$$

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However, $L^p(m)$, $p > 1$, is not reflexive whenever $L^1(m) \neq L^1_w(m)$.

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

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$$(L^{p_0}(m), L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m), L^{p_1}(m))_{\theta,q} = (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta,q} = L^{p,q}(\|m\|)$$

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Lorent space $L^{p,q}(\|m\|)$ with respect to $\|m\|$: $1 \leq p < \infty$, $0 < q \leq \infty$, space of (m-a.e equivalence classes of) scalar measurable functions on Ω s.t.

$$\|f\|_{L^{p,q}(\|m\|)} := \begin{cases} \left(\int_0^\infty \left[s^{\frac{1}{p}} f_*(s) \right]^q \frac{ds}{s} \right)^{\frac{1}{q}}, & \text{for} \quad 1 \leq q < \infty, \\ \sup\limits_{s>0} s^{\frac{1}{p}} f_*(s), & \text{for} \quad q = \infty, \end{cases}$$

is finite.

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is finite. Here f_* is the decreasing rearrangement (with respect to m) of f

$$f_*(t) := \inf\{s > 0 : \|m\|(\{w \in \Omega : |f(w)| > s\}) \leq t\},$$

and $\|m\|(A) := \sup\{|\langle m, x^*\rangle|(A) : x^* \in B(X^*)\}$ the semivariation of m.

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$L^0(m)$ denotes the space of all measurable functions $f : \Omega \to \mathbb{C}$. Two functions $f, g \in L^0(m)$ will be identified if are equal m–a.e., that is, whenever

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• $f \in L^0(m)$ is called **integrable** (with respect to m) if

i) $f \in L^1(|\langle m, x^*\rangle|)$, for all $x^* \in X^*$ (i.e. f is **weakly integrable**)

ii) given any $A \in \Sigma$, there exists an element $\int_A f dm \in X$ such that $\left\langle \int_A f dm, x^* \right\rangle = \int_A f \, d\langle m, x^*\rangle$, for all $x^* \in X^*$.

Let

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endowed with the norm

$$\|f\|_1 := \sup\left\{ \int_\Omega |f| \, d\, |\langle m, x^*\rangle| : x^* \in B(X^*) \right\}.$$

- Given $1 < p < \infty$, $f \in L^0(m)$ is said to be
  i) **weakly p-integrable** (with respect to m) if $|f|^p \in L^1_w(m)$,
  ii) **p-integrable** (with respect to m) if $|f|^p \in L^1(m)$,

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- Some properties:
  - $L^p(m)$ is a Banach lattice with order continuous norm.
  - $L^p_w(m)$ is a Banach lattice with the Fatou property.
  - $L^p(m)$ y $L^p_w(m)$ may not be reflexive for $p > 1$.
  - If $1 < p_1 < p_2 < \infty$, then
    $L^\infty(m) \subseteq L^{p_2}(m) \subseteq L^{p_2}_w(m) \subseteq L^{p_1}(m) \subseteq L^{p_1}_w(m) \subseteq L^1(m) \subseteq L^1_w(m)$.

• Given a Banach couple $\bar{X} = (X_0, X_1)$, let $\mathcal{F}(\bar{X})$ be the set of functions f defined on $S = \{z \in \mathbb{C} : 0 \leq \text{ Re } z \leq 1\}$ with values in $X_0 + X_1$, s.t.:

(a) f is bounded and continuous on S, and analytic on the open strip $S_0 = \{z \in \mathbb{C} : 0 < \text{ Re } z < 1\}$,

(b) the functions $t \longrightarrow f(j + it)$ $(j = 0, 1)$ are continuous from $\mathbb{R}$ into X_j, and tend to zero as $|t| \to \infty$.

The norm considered in $\mathcal{F}(\bar{X})$ is

$$\|f\|_{\mathcal{F}} = \max\Big\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1} \Big\}.$$

The **complex interpolation space** $[X_0, X_1]_{[\theta]}$, $0 < \theta < 1$, consists in

$$x \in X_0 + X_1 \text{ such that } x = f(\theta), \text{ for some } f \in \mathcal{F}(\bar{X}).$$

$[X_0, X_1]_{[\theta]}$ is a Banach space with the norm

$$\|x\|_{[\theta]} = \inf \Big\{ \|f\|_{\mathcal{F}} : f(\theta) = x,\ f \in \mathcal{F}(\bar{X}) \Big\}.$$

• Given a Banach couple $\bar{X} = (X_0, X_1)$, let $\mathcal{G}(\bar{X})$ be the set of functions g defined on $S = \{z \in \mathbb{C} : 0 \leq \text{ Re } z \leq 1\}$ with values in $X_0 + X_1$, s.t.:

(a) $\|g(z)\|_{X_0+X_1} \leq c(1 + |z|)$,

(b) g is continuous on S and analytic on $S_0 = \{z \in \mathbb{C} : 0 < \text{ Re } z < 1\}$,

(c) the function $g(j + it_1) - g(j + it_2)$ has values in X_j, for all real values of t_1 and t_2 and for $j = 0, 1$, and

$$\|g\|_{\mathcal{G}} = \max\left\{ \sup_{t_1, t_2 \in \mathbb{R}} \left\| \frac{g(it_1) - g(it_2)}{t_1 - t_2} \right\|_{X_0}, \left\| \frac{g(1 + it_1) - g(1 + it_2)}{t_1 - t_2} \right\|_{X_1} \right\}.$$

The space $\mathcal{G}(\bar{X})$, reduced modulo constant functions and provided with the norm $\|g\|_{\mathcal{G}}$, is a Banach space.

The **complex interpolation space** $[X_0, X_1]^{[\theta]}$, $0 < \theta < 1$, consists in

$$x \in X_0 + X_1 \text{ such that } x = g'(\theta), \text{ for some } g \in \mathcal{G}(\bar{X}).$$

$[X_0, X_1]^{[\theta]}$ is a Banach space with the norm

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$$[L^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L_w^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} = L^p(m),$$

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R. Campo, A. Fernández, A. M., F. Mayoral and F. Naranjo, *Complex interpolation of Orlicz spaces with respect to a vector measure*, Math. Nachr. **287** (2014), 23–31.

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• Given $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$, $\phi^{-1} = (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$, do the following equalities hold?

$$[L^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m),$$

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- An N-function is any function $\phi : [0,\infty) \to [0,\infty)$ which is
  - strictly increasing,
  - continuous,
  - convex,
  - $\phi(0) = 0$,
  - $\lim_{x\to 0} \frac{\phi(x)}{x} = 0$,
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An N-function has the Δ_2-property (we write $\phi \in \Delta_2$) if

$$\exists C > 0 \text{ such that } \phi(2x) \le C\phi(x) \text{ for all } x \ge 0.$$

- The **weak Orlicz space** $L_w^{\phi}(m)$ (w.r.t. m and ϕ) is defined as

$$L_w^{\phi}(m) := \left\{ f \in L^0(m) : \|f\|_{L_w^{\phi}(m)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{L_w^{\phi}(m)} &:= \sup \left\{ \|f\|_{L^{\phi}(|\langle m, x^* \rangle|)} : x^* \in B_{X^*} \right\} \\ &= \sup_{x^* \in B_{X^*}} \inf \left\{ k > 0 : \int_{\Omega} \phi\left(\frac{|f|}{k} \right) d|\langle m, x^* \rangle| \leq 1 \right\}. \end{aligned}$$

- The **weak Orlicz space** $L^{\phi}_{w}(m)$ (w.r.t. m and ϕ) is defined as

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$L^{\phi}_{w}(m)$ coincides with the intersection of all Orlicz $L^{\phi}(|\langle m, x^* \rangle|)$, $x^* \in X^*$.

- The **weak Orlicz space** $L_w^\phi(m)$ (w.r.t. m and ϕ) is defined as

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- The **Orlicz space** $L^\phi(m)$ (w.r.t. m and ϕ) is defined by $\overline{\mathcal{S}(\Sigma)}^{L_w^\phi(m)}$.

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- If $\phi(x) = x^p$, $L_w^\phi(m)$ and $L^\phi(m)$ correspond to $L_w^p(m)$ and $L^p(m)$, respect.

- The **weak Orlicz space** $L^{\phi}_{w}(m)$ (w.r.t. m and ϕ) is defined as

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- The **Orlicz space** $L^{\phi}(m)$ (w.r.t. m and ϕ) is defined by $\overline{\mathcal{S}(\Sigma)}^{L^{\phi}_{w}(m)}$.
- If $\phi(x) = x^p$, $L^{\phi}_{w}(m)$ and $L^{\phi}(m)$ correspond to $L^p_w(m)$ and $L^p(m)$, respect.
- The corresponding **Orlicz classes** (w.r.t. m and ϕ) are given by

$$O^{\phi}_{w}(m) := \{f :\in L^0(m) : \phi(|f|) \in L^1_w(m)\},$$

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$$L^\phi_w(m) := \left\{ f \in L^0(m) : \|f\|_{L^\phi_w(m)} < \infty \right\},$$

where

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When $\phi \in \Delta_2$

$$O_w^\phi(m) = L_w^\phi(m) \text{ and } O^\phi(m) = L^\phi(m).$$

- Let (X_0, X_1) be a couple of Banach lattices on the same measure space and $0 < \theta < 1$, the **Calderón's space** $X_0^{1-\theta} X_1^{\theta}$ is

$$X_0^{1-\theta} X_1^{\theta} := \{f \in L^0 : \exists \lambda > 0, \exists f_i \in B_{X_i} \text{ s.t. } |f| \leq \lambda |f_0|^{1-\theta} |f_1|^{\theta}\},$$

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It holds that

CL1 $X_0 \cap X_1 \subseteq [X_0, X_1]_{[\theta]} \subseteq X_0^{1-\theta}X_1^{\theta} \subseteq [X_0, X_1]^{[\theta]} \subseteq X_0 + X_1.$

CL2 If X_0 or X_1 is order continuous, then $[X_0, X_1]_{[\theta]} = X_0^{1-\theta}X_1^{\theta}$.

CL3 If X_0 and X_1 have the Fatou property then $[X_0, X_1]^{[\theta]} = X_0^{1-\theta}X_1^{\theta}$.

- Given a Banach couple (X_0, X_1) and $0 < \theta < 1$, the **Gustavsson-Peetre space** $\langle X_0, X_1, \theta\rangle$ is the Banach space formed by

$$x \in X_0 + X_1 \text{ for which } \exists (x_k)_{k\in\mathbb{Z}} \subseteq X_0 \cap X_1 \text{ s.t.}$$

GP1 $x = \sum_{k\in\mathbb{Z}} x_k$, where the series converges in $X_0 + X_1$.

GP2 $\exists C > 0$ s.t. for every finite subset $F \subseteq \mathbb{Z}$ and every subset of scalars $(\varepsilon_k)_{k\in F}$, with $|\varepsilon_k| \leq 1$,

$$\left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} x_k \right\|_{X_0} \leq C \quad \text{and} \quad \left\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} x_k \right\|_{X_1} \leq C.$$

The norm considered in $\langle X_0, X_1, \theta\rangle$ is

$$\|x\|_{\langle X_0, X_1, \theta\rangle} = \inf\{C > 0 : \text{taken over all } (x_k)_{k\in\mathbb{Z}} \text{ satisfying GP1 and GP2}\}.$$

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Moreover, $\langle X_0, X_1, \theta\rangle \subseteq [X_0, X_1]^{[\theta]}$.

Proposition

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. Then

(1) $L^{\phi_0}(m)^{1-\theta} L^{\phi_1}(m)^{\theta} = L^{\phi}(m)$.

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PROPOSITION

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Corollary

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and ϕ s.t. $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. It holds that

$$[L^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m).$$

$$[L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$$

M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., 1991.

- Some partial ordering relations between N-functions:

  $\phi_1 \prec \phi_0$ if $\exists\varepsilon > 0$, $\exists x_0 \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_0$.

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Lemma

Let $\phi_0, \phi_1 \in \Delta_2$.

(1) If $\phi_1 \prec \phi_0$, then $L_w^{\phi_0}(m) \subseteq L_w^{\phi_1}(m)$, and $L^{\phi_0}(m) \subseteq L^{\phi_1}(m)$.

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For $\phi_1(x) := x^p$, $\phi_0(x) := x^q$, $1 < p < q$, it follows that $\phi_1 \prec\!\!\prec \phi_0$, and therefore the well-known inclusion $L_w^q(m) \subseteq L^p(m)$.

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  $\phi_1 \lll \phi_0$ if $\forall\varepsilon > 0$, $\exists x_\varepsilon \geq 0$ s.t. $\phi_1(x) \leq \phi_0(\varepsilon x)$, for all $x \geq x_\varepsilon$.

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Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$.

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Theorem

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$.
If $\phi_1 \lll \phi_0$, it follows that

$$\langle L^{\phi_0}(m), L^{\phi_1}(m), \theta \rangle = L_w^{\phi}(m).$$

Lemma

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. There exits a continuous function $h : (0, \infty) \to (0, \infty)$ s.t.

$$\phi(z) = \phi_0(zh(z)^{-\theta}) = \phi_1(zh(z)^{1-\theta}), \quad z > 0.$$

Moreover, if $\phi_1 \lll \phi_0$ then $\lim_{z\to\infty} h(z) = \infty$.

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Sketch of the proof of the last theorem: $\langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle = L^{\phi}_w(m)$.

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Given any $x^* \in X^*$, it follows that

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which gives $\langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq L^{\phi}_w(m)$.

Now, let $f \in L^{\phi}_w(m)$, $f \geq 0$, $\|f\|_{L^{\phi}_w(m)} \leq 1$ and consider the function h given by the previous lemma. Fix $k \in \mathbb{Z}$, let $f_k := f\chi_{A_k}$ where

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f_k is a bounded function. If not, $\exists (w_n) \subseteq A_k$ s.t. $f(w_n) \to \infty$ and then $h(f(w_n)) \to \infty$, which contradicts $(w_n) \subseteq A_k$.

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Let us check that $(f_k)_{k\in\mathbb{Z}}$ satisfies GP1 and GP2.

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$$\phi_0\left(\left|\sum_{k\in F}\frac{\varepsilon_k}{2^{k\theta}}f_k\right|\right) \le \sum_{k\in F}\phi_0(2^{-k\theta}f_k) \le \sum_{k\in F}\phi_0(2h(f_k)^{-\theta}f_k)$$

$$\le \quad C\sum_{k\in F}\phi_0(h(f_k)^{-\theta}f_k) = \sum_{k\in F}\phi(f_k) = C\phi\left(\sum_{k\in F}f_k\right) \le C\phi(f)$$

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GP2. Let $F \subseteq \mathbb{Z}$ finite and $(\varepsilon_k)_{k\in F}$ with $|\varepsilon_k| \le 1$. Using the definitions of functions f_k and h, and $\phi_0 \in \Delta_2$,

$$\phi_0\left(\left|\sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} f_k\right|\right) \le \sum_{k\in F} \phi_0(2^{-k\theta} f_k) \le \sum_{k\in F} \phi_0(2h(f_k)^{-\theta} f_k)$$

$$\le \quad C \sum_{k\in F} \phi_0(h(f_k)^{-\theta} f_k) = \sum_{k\in F} \phi(f_k) = C\phi\left(\sum_{k\in F} f_k\right) \le C\phi(f)$$

Then $\left\|\phi_0\left(\left|\sum_{k\in F} \varepsilon_k f_k/2^{k\theta}\right|\right)\right\|_{L^1_w(m)} \le C\|\phi(f)\|_{L^1_w(m)} \le C$. Hence, it can be proved that $\left\|\sum_{k\in F} \varepsilon_k f_k/2^{k\theta}\right\|_{L^{\phi_0}_w(m)} \le C+1$.

Let us check that $(f_k)_{k\in\mathbb{Z}}$ satisfies GP1 and GP2.

GP1. Evidently $f = \sum_{k=-\infty}^{\infty} f_k$ pointwise, each of its partial sums is pointwise bounded by $f \in L_w^{\phi}(m) \subseteq L^{\phi_1}(m)$. The order continuity of the norm in $L^{\phi_1}(m)$ gives the convergence of $\sum_{k=-\infty}^{\infty} f_k$ to f in $L^{\phi_1}(m) = L^{\phi_0}(m) + L^{\phi_1}(m)$.

GP2. Let $F \subseteq \mathbb{Z}$ finite and $(\varepsilon_k)_{k\in F}$ with $|\varepsilon_k| \leq 1$. Using the definitions of functions f_k and h, and $\phi_0 \in \Delta_2$,

$$\phi_0\left(\left|\sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} f_k\right|\right) \leq \sum_{k\in F} \phi_0(2^{-k\theta} f_k) \leq \sum_{k\in F} \phi_0(2h(f_k)^{-\theta} f_k)$$

$$\leq \quad C\sum_{k\in F} \phi_0(h(f_k)^{-\theta} f_k) = \sum_{k\in F} \phi(f_k) = C\phi\left(\sum_{k\in F} f_k\right) \leq C\phi(f)$$

Then $\left\|\phi_0\left(\left|\sum_{k\in F} \varepsilon_k f_k/2^{k\theta}\right|\right)\right\|_{L_w^1(m)} \leq C\|\phi(f)\|_{L_w^1(m)} \leq C$. Hence, it can be proved that $\left\|\sum_{k\in F} \varepsilon_k f_k/2^{k\theta}\right\|_{L_w^{\phi_0}(m)} \leq C+1$.

Similar arguments yield that $\left\|\sum_{k\in F} \varepsilon_k f_k/2^{k(\theta-1)}\right\|_{L_w^{\phi_1}(m)} \leq 1$. $\square$

$L_w^{\phi}(m)$

$$L_w^{\phi}(m) \quad = \quad \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta \rangle$$

$$L^{\phi}_{w}(m) \quad = \quad \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta \rangle \ \subseteq \ [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]}$$

$$
\begin{aligned}
L^{\phi}_{w}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta \rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\
&\subseteq [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]}
\end{aligned}
$$

$$\begin{aligned} L_w^{\phi}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = (L_w^{\phi_0}(m))^{1-\theta}(L_w^{\phi_1}(m))^{\theta} \end{aligned}$$

$$
\begin{aligned}
L^{\phi}_{w}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\
&\subseteq [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = (L^{\phi_0}_{w}(m))^{1-\theta}(L^{\phi_1}_{w}(m))^{\theta} = L^{\phi}_{w}(m).
\end{aligned}
$$

$$\begin{aligned} L^{\phi}_{w}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = (L^{\phi_0}_{w}(m))^{1-\theta}(L^{\phi_1}_{w}(m))^{\theta} = L^{\phi}_{w}(m). \end{aligned}$$

Therefore,

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = L^{\phi}_{w}(m),$$

$$\begin{aligned} L_w^{\phi}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = (L_w^{\phi_0}(m))^{1-\theta}(L_w^{\phi_1}(m))^{\theta} = L_w^{\phi}(m). \end{aligned}$$

Therefore,

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m),$$

and, by $L^{\phi_i}(m) \subseteq L_w^{\phi_i}(m)$ $(i = 0, 1)$, it also holds that

$$[L^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$$

$$\begin{aligned} L^{\phi}_{w}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = (L^{\phi_0}_{w}(m))^{1-\theta}(L^{\phi_1}_{w}(m))^{\theta} = L^{\phi}_{w}(m). \end{aligned}$$

Therefore,

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = L^{\phi}_{w}(m),$$

and, by $L^{\phi_i}(m) \subseteq L^{\phi_i}_{w}(m)$ $(i = 0, 1)$, it also holds that

$$[L^{\phi_0}(m), L^{\phi_1}_{w}(m)]^{[\theta]} = [L^{\phi_0}_{w}(m), L^{\phi_1}(m)]^{[\theta]} = L^{\phi}_{w}(m).$$

This gives (i) in the following theorem.

$$\begin{aligned} L_w^{\phi}(m) &= \langle L^{\phi_0}(m), L^{\phi_1}(m), \theta\rangle \subseteq [L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} \\ &\subseteq [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = (L_w^{\phi_0}(m))^{1-\theta}(L_w^{\phi_1}(m))^{\theta} = L_w^{\phi}(m). \end{aligned}$$

Therefore,

$$[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m),$$

and, by $L^{\phi_i}(m) \subseteq L_w^{\phi_i}(m)$ ($i = 0, 1$), it also holds that

$$[L^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$$

This gives (i) in the following theorem.

Theorem

Let $\phi_0, \phi_1 \in \Delta_2$, $0 < \theta < 1$ and let ϕ be given by $\phi^{-1} := (\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}$. If $\phi_1 \lll \phi_0$, then

(i) $[L^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = [L^{\phi_0}(m), L_w^{\phi_1}(m)]^{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]^{[\theta]} = L_w^{\phi}(m).$

(ii) $[L_w^{\phi_0}(m), L_w^{\phi_1}(m)]_{[\theta]} = [L^{\phi_0}(m), L_w^{\phi_1}(m)]_{[\theta]} = [L_w^{\phi_0}(m), L^{\phi_1}(m)]_{[\theta]} = L^{\phi}(m).$

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