On the minimal property of de la Vallée Poussin's operator

Beata Deregowska, Barbara Lewandowska

7 lipca 2015

Definition

Let X be a Banach space and V be a linear, closed subspace of X. Then by $\mathcal{P}(X, V)$ we denote the set of all linear projections continuous with respect to the operator norm. An operator $P : X \to V$ is called a projection, if $P|_V = id_V$. A projection $P_0 \in \mathcal{P}(X, V)$ is called minimal if

$$\|P_0\| = \lambda(V, X) := \inf\{\|P\| : P \in \mathcal{P}(X, V)\}.$$

For all $x \in X$

$$\|x - Px\| \leq \|Id - P\| \cdot \mathrm{dist}(x, V) \leq (1 + \|P\|) \cdot \mathrm{dist}(x, V),$$

where $\mathrm{dist}(x, V) := \inf\{\|x - v\| : v \in V\}$.

Definition

Let $V \subset Z$ be two subspaces of a Banach space X. Then

$$\mathcal{P}_V(X,Z) := \{P \in \mathcal{L}(X,Z) : P|_V = id\}.$$

An element $P_o \in \mathcal{P}_V(X,Z)$ is called a *minimal generalized projection* (MGP) if

$$\|P_o\| = \lambda_Z(V,X) := \inf\{\|P\| : P \in \mathcal{P}_V(X,Z)\}$$

Definition

Let $\mathcal{C}_0(2\pi)$ denote the space of all continuous, 2π-periodic functions equipped with the supremum norm. Let Π_n denote the space of all trigonometric polynomials of degree less than or equal to n. The Fourier projection $F_n : \mathcal{C}_0(2\pi) \to \Pi_n$ is defined by

$$F_n(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(s) D_n(t-s) ds,$$

where D_n is the Dirichlet kernel

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}.$$

It is well known by the classical result of Lozinski [1] that Fourier operator F_n has the minimal norm among all projections from $\mathcal{C}_0(2\pi)$ onto Π_n. If we replace $\mathcal{C}_0(2\pi)$ by $L_1[0,2\pi]$ the Lozinski theorem stays true. In 1969, Cheney, Hobby, Morris, Schurer and Wulbert [2] have proved that the Fourier projection is the unique minimal with respect to the operator norm in $\mathcal{L}(\mathcal{C}_0(2\pi))$. In the same year, Lambert [3] proved the analogous result for $L_1[0,2\pi]$.

[1] S. M. Lozinski, *On a class of linear operators,* Dokl. Acad. Nauk SSSR, Vol. 61, No. 2 (1948), 193-196;

[2] E. W. Cheney, C. R. Hobby, P. D. Morris, F. Schurer and D. E. Wulbert, *On the minimal property of the Fourier projection,* Trans. Amer. Math. Soc. Vol. 143 (1969), 249-258;

[3] P. V. Lambert, *On the minimum norm property of the Fourier projection in L_1 -spaces,* Bull. Soc. Math. Belg. 21. (1969), 370-391.

Definition

De la Vallée Poussin's operator $H_n : X \to \Pi_{2n-1}$ is given by

$$H_n(f)(t) = \frac{1}{n}\sum_{k=n}^{2n-1} F_k(f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s)V_n(t-s)ds,$$

where

$$V_n(t) = \frac{1}{n}\sum_{k=n}^{2n-1} D_n(t) = \frac{(\sin(nx))^2 - (\sin(\frac{n}{2}x))^2}{n(\sin(\frac{1}{2}x))^2}.$$

Theorem

Let X be $\mathcal{C}_0(2\pi)$ or $L_1[0, 2\pi]$. Then de la Vallée Poussin's operator H_n is a minimal generalized projection in $\mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$.

Lemma

Let X be $\mathcal{C}_0(2\pi)$ or $L_1[0, 2\pi]$. For every $P \in \mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$ there exists $\widetilde{P} \in \mathcal{P}_{\Pi_n}(X, \Pi_{2n-1})$ such that

$$\|\widetilde{P}\| \leq \|P\| \quad \textit{and} \quad \widetilde{P}(f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(s) \sum_{k=-2n+1}^{2n-1} a_k e^{ik(t-s)} ds$$

for some $a_k \in \mathbb{C}$.

It is worth mentioning that Mehta showed that the norm of the de la Vallée Poussin operator is always equal $\frac{1}{3} + \frac{2\sqrt{3}}{\pi}$.

H. Mehta, *The L1 norm of the generalized de la Vallee Poussin kernel,* arXiv:1311.1407 [math.CA] (2013), 1-12.

Definition

Let $X = \mathcal{C}_0(2\pi)$ or $X = L_1[0, 2\pi]$. Let $H_{rn,sn} : X \to \Pi_{sn-1}$ be given by

$$H_{rn,sn}(f)(t) = \frac{1}{(s-r)n} \sum_{k=rn}^{sn-1} F_k(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(s) V_{rn,sn}(t-s) ds, \quad (1)$$

where

$$V_{rn,sn}(t) = \frac{1}{(s-r)n} \sum_{k=rn}^{sn-1} D_k(t) = \frac{\sin(\frac{s-r}{2} nt) \sin(\frac{s+r}{2} nt)}{(s-r)n(\sin(\frac{1}{2}t))^2}. \quad (2)$$

for some $r, s \in \mathbb{N}$ such that $r < s$.
Then $H_{rn,sn}$ is called de la Vallée Poussin's type operator.

Observe that for every operator P which is de la Vallée Poussin's type operator there exist unique natural numbers r, s, n such that r and s are coprime and $P = H_{rn,sn}$.

Theorem

Let $X = \mathcal{C}_0(2\pi)$ or $X = L_1[0, 2\pi]$. Then for all $r \in \mathbb{N}$ operator $H_{rn,(r+1)n}$ is MGP in $\mathcal{P}_{\Pi_{rn}}(X,\ \Pi_{(r+1)n-1})$.

Theorem

Let $X = \mathcal{C}_0(2\pi)$ or $X = L_1[0, 2\pi]$. Then for all $r \in \mathbb{N}$ operator $H_{rn,(r+2)n}$ is MGP in $\mathcal{P}_{\Pi_{rn}}(X,\ \Pi_{(r+2)n-1})$.

Theorem

Let $n, s \in \mathbb{N}$ and $s > 3$. Then $H_{n,sn}$ is not a minimal generalized projection in $\mathcal{P}_{\Pi_n}(X, \Pi_{sn-1})$.

Hypothesis

Let $X = \mathcal{C}_0(2\pi)$ or $X = L_1[0, 2\pi]$. If r, s are coprime and $H_{rn,sn}$ is MGP in $\mathcal{P}_{\Pi_{rn}}(X, \Pi_{sn-1})$ then $s = r + 1$ or $s = r + 2$.

For more details, see: B. Deregowska, B. Lewandowska, *On the minimal property of de la Vallée Poussin's operator,* Bull. Aust. Math. Soc. 91 (2015), no. 1, 129-133.

Thank you for your attention