INTERPOLATION OF CESÁRO AND TANDORI SPACES

Karol Leśnik

Poznań University of Technology

Function Spaces XI 2015

Cesàro operator

For $f \in L^1_{loc}$

$$Cf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty. \tag{1}$$

Copson operator

For $f \in L^1_{loc}$

$$C^*f(x) = \int_x^\infty \frac{f(t)}{t}dt, \quad 0 < x < \infty. \tag{2}$$

Nonincreasing majorant

For $f \in L^0$

$$\tilde{f}(x) = \operatorname*{ess\,sup}_{x \leqslant t} |f(t)|, \quad 0 < x < \infty. \tag{3}$$

Cesàro operator

For $f \in L^1_{loc}$

$$Cf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty. \tag{1}$$

Copson operator

For $f \in L^1_{loc}$

$$C^*f(x) = \int_x^\infty \frac{f(t)}{t}dt, \quad 0 < x < \infty. \tag{2}$$

Nonincreasing majorant

For $f \in L^0$

$$\tilde{f}(x) = \operatorname*{ess\,sup}_{x \leqslant t} |f(t)|, \quad 0 < x < \infty. \tag{3}$$

Cesàro operator

For $f \in L^1_{loc}$

$$Cf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty. \tag{1}$$

Copson operator

For $f \in L^1_{loc}$

$$C^*f(x) = \int_x^\infty \frac{f(t)}{t}dt, \quad 0 < x < \infty. \tag{2}$$

Nonincreasing majorant

For $f \in L^0$

$$\tilde{f}(x) = \operatorname*{ess\,sup}_{x \leqslant t} |f(t)|, \quad 0 < x < \infty. \tag{3}$$

Banach function space (B.f.s.)

A Banach space $X \subset L^0(\mathbb{R}_+)$ with

- if $x \in X, y \in L^0$ and $|y| \leqslant |x|$ -a.e., then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$
- there is $x \in X$ such that $x(t) > 0$ a.e.

Cesàro space

For a B.f.s. X

$$CX = \{f \in L^0 : \|f\|_{CX} = \|C|f|\|_X < \infty\}. \tag{4}$$

Tandori space

For a B.f.s. X

$$\widetilde{X} = \{f \in L^0 : \|f\|_{\widetilde{X}} = \|\tilde{f}\|_X < \infty\}. \tag{5}$$

Banach function space (B.f.s.)

A Banach space $X \subset L^0(\mathbb{R}_+)$ with

- if $x \in X, y \in L^0$ and $|y| \leqslant |x|$ -a.e., then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$
- there is $x \in X$ such that $x(t) > 0$ a.e.

Cesàro space

For a B.f.s. X

$$CX = \{f \in L^0 : \|f\|_{CX} = \|C|f|\|_X < \infty\}. \tag{4}$$

Tandori space

For a B.f.s. X

$$\widetilde{X} = \{f \in L^0 : \|f\|_{\widetilde{X}} = \|\tilde{f}\|_X < \infty\}. \tag{5}$$

Banach function space (B.f.s.)

A Banach space $X \subset L^0(\mathbb{R}_+)$ with

- if $x \in X, y \in L^0$ and $|y| \leqslant |x|$ -a.e., then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$
- there is $x \in X$ such that $x(t) > 0$ a.e.

Cesàro space

For a B.f.s. X

$$CX = \{f \in L^0 : \|f\|_{CX} = \|C|f|\|_X < \infty\}. \tag{4}$$

Tandori space

For a B.f.s. X

$$\widetilde{X} = \{f \in L^0 : \|f\|_{\widetilde{X}} = \|\tilde{f}\|_X < \infty\}. \tag{5}$$

Ecxample

- For $X = L^p$ with $p > 1$ we have $CL^p = Ces_p$
- For $p > 1$ we have $Ces_p' = \widetilde{L^{p'}}$

Theorem (Maligranda-KL 2015)

Let X be a B.f.s. with the Fatou property such that $C : X \to X$ is bounded. If the dilation operator σ_b is bounded on X for some $0 < b < 1$ then

$$(CX)' = \widetilde{X'}$$

.

Ecxample

- For $X = L^p$ with $p > 1$ we have $CL^p = Ces_p$
- For $p > 1$ we have $Ces_p' = \widetilde{L^{p'}}$

Theorem (Maligranda-KL 2015)

Let X be a B.f.s. with the Fatou property such that $C : X \to X$ is bounded. If the dilation operator σ_b is bounded on X for some $0 < b < 1$ then

$$(CX)' = \widetilde{X'}$$

.

Interpolation spaces

- X is **intermediate** for a (compatible) couple of Banach spaces (X_0, X_1) when $X_0 \cap X_1 \subset X \subset X_0 + X_1$.
- $T : (X_0, X_1) \to (X_0, X_1)$ when T is defined on $X_0 + X_1$ and

$$T : X_0 \to X_0 \text{ and } T : X_1 \to X_1$$

  with

$$\|T\|_{(X_0,X_1)\to(X_0,X_1)} = \max\{\|T\|_{X_0\to X_0}, \|T\|_{X_1\to X_1}\}.$$

- X is **interpolation** space for the couple (X_0, X_1) (we write $X \in \mathrm{int}(X_0, X_1)$) when X is intermediate and for each $T : (X_0, X_1) \to (X_0, X_1)$ there holds

$$T : X \to X.$$

Interpolation spaces

- X is **intermediate** for a (compatible) couple of Banach spaces (X_0, X_1) when $X_0 \cap X_1 \subset X \subset X_0 + X_1$.
- $T : (X_0, X_1) \to (X_0, X_1)$ when T is defined on $X_0 + X_1$ and

$$T : X_0 \to X_0 \text{ and } T : X_1 \to X_1$$

with

$$\|T\|_{(X_0,X_1)\to(X_0,X_1)} = \max\{\|T\|_{X_0\to X_0}, \|T\|_{X_1\to X_1}\}.$$

- X is **interpolation** space for the couple (X_0, X_1) (we write $X \in \mathrm{int}(X_0, X_1)$) when X is intermediate and for each $T : (X_0, X_1) \to (X_0, X_1)$ there holds

$$T : X \to X.$$

Interpolation spaces

- X is **intermediate** for a (compatible) couple of Banach spaces (X_0, X_1) when $X_0 \cap X_1 \subset X \subset X_0 + X_1$.
- $T : (X_0, X_1) \to (X_0, X_1)$ when T is defined on $X_0 + X_1$ and

$$T : X_0 \to X_0 \text{ and } T : X_1 \to X_1$$

with

$$\|T\|_{(X_0,X_1)\to(X_0,X_1)} = \max\{\|T\|_{X_0\to X_0}, \|T\|_{X_1\to X_1}\}.$$

- X is **interpolation** space for the couple (X_0, X_1) (we write $X \in \mathrm{int}(X_0, X_1)$) when X is intermediate and for each $T : (X_0, X_1) \to (X_0, X_1)$ there holds

$$T : X \to X.$$

K-functional

For $f \in X_0 + X_1$ the K-**functional** of f with respect to the couple (X_0, X_1) is defined as

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\} \ \text{ for } t > 0.$$

K-method of interpolation

For a given B.f.s. E over containing the function $\min\{t, 1\}$ define

$$(X_0, X_1)_E^K = \{f \in X_0 + X_1 : K(\cdot, f, X_0, X_1) \in E\}$$

with the norm

$$\|f\|_{(X_0,X_1)_E^K} = \|K(\cdot, f, X_0, X_1)\|_E.$$

K-functional

For $f \in X_0 + X_1$ the K-**functional** of f with respect to the couple (X_0, X_1) is defined as

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\} \text{ for } t > 0.$$

K-method of interpolation

For a given B.f.s. E over containing the function $\min\{t, 1\}$ define

$$(X_0, X_1)_E^K = \{f \in X_0 + X_1 : K(\cdot, f, X_0, X_1) \in E\}$$

with the norm

$$\|f\|_{(X_0,X_1)_E^K} = \|K(\cdot, f, X_0, X_1)\|_E.$$

Calderón couple

A couple (X_0, X_1) is called Calderón couple when each interpolation space may be generated by the K-method.

Theorem [Brudnyi-Kruglyak]

TFAE:

i) For $X \in \text{int}(X_0, X_1)$ there is a B.f.s. E such that

$$(X_0, X_1)^K_E = X$$

ii) For each $f, g \in X_0 + X_1$

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1) \implies \exists_{T:(X_0,X_1)\to(X_0,X_1)} Tg = f$$

iii) For each $f, g \in X_0 + X_1$ and $X \in \text{int}(X_0, X_1)$

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1), g \in X \implies f \in X$$

Calderón couple

A couple (X_0, X_1) is called Calderón couple when each interpolation space may be generated by the K-method.

Theorem [Brudnyi-Kruglyak]

TFAE:

i) For $X \in \operatorname{int}(X_0, X_1)$ there is a B.f.s. E such that

$$(X_0, X_1)_E^K = X$$

ii) For each $f, g \in X_0 + X_1$

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1) \implies \exists_{T:(X_0,X_1)\to(X_0,X_1)} Tg = f$$

iii) For each $f, g \in X_0 + X_1$ and $X \in \operatorname{int}(X_0, X_1)$

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1), g \in X \implies f \in X$$

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Calderón couples:

- (L^1, L^∞) - Calderón
- (L^1, L^q) and (L^p, L^∞) - Lorentz-Shimogaki
- (L^p, L^q) - Cwikel, Arazy-Cwikel, Sparr
- (E, L^∞) for symmetric E which is strechable - Kalton
- (H^p, H^q) - Jones
- $((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p_1})$ - Cwikel
- (Ces_∞, L^1) - Mastyło-Sinnamon

Non Calderón couples:

- $(L^1 + L^\infty, L^1 \cap L^\infty)$ - Maligranda-Ovchinnikov
- $(C[0,1], \Lambda_\theta[0,1])$ - Cwikel-Mastyło
- (E, L^∞) for non-strechable symmetric E - Kalton

Sinnamon's question 2006

Is a dual couple of (Ces_∞, L^1), i.e. $(\widetilde{L^1}, L^\infty)$ also a Calderón couple?

Theorem (KL 2015)

The couple $(\widetilde{L^1}, L^\infty)$ is a Calderón couple.

Sinnamon's question 2006

Is a dual couple of (Ces_∞, L^1), i.e. $(\widetilde{L^1}, L^\infty)$ also a Calderón couple?

Theorem (KL 2015)

The couple $(\widetilde{L^1}, L^\infty)$ is a Calderón couple.

Sketch of proof

Let $f, g \in \widetilde{L^1} + L^\infty$ be such that

$$K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty).$$

We need to find $H : (\widetilde{L^1}, L^\infty) \to (\widetilde{L^1}, L^\infty)$ such that

$$Hg = f$$

Scheme:

$$g \xmapsto{\quad S \quad} \tilde{g} \xmapsto{\quad T \quad} \tilde{f} \xmapsto{\quad M \quad} f$$

$$(\widetilde{L^1}, L^\infty) \xrightarrow{\ S\ } (\widetilde{L^1}, L^\infty) \xrightarrow{\ T\ } (\widetilde{L^1}, L^\infty) \xrightarrow{\ M\ } (\widetilde{L^1}, L^\infty)$$

Easy part:

$$M = M_{f/\tilde{f}}.$$

Sketch of proof

Let $f, g \in \widetilde{L^1} + L^\infty$ be such that

$$K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty).$$

We need to find $H : (\widetilde{L^1}, L^\infty) \to (\widetilde{L^1}, L^\infty)$ such that

$$Hg = f$$

Scheme:

$$g \xmapsto{\quad S \quad} \tilde{g} \xmapsto{\quad T \quad} \tilde{f} \xmapsto{\quad M \quad} f$$

$$(\widetilde{L^1}, L^\infty) \xrightarrow{S} (\widetilde{L^1}, L^\infty) \xrightarrow{T} (\widetilde{L^1}, L^\infty) \xrightarrow{M} (\widetilde{L^1}, L^\infty)$$

Easy part:

$$M = M_{f/\tilde{f}}.$$

Looking for S

Proposition

Let $g \in \widetilde{L^0} = \{f \in L^0 : \tilde{f} \in L^0\}$. Then for each $q > 1$ there is a linear operator S defined on $\widetilde{L^0}$ such that

$$Sg = \widetilde{g}$$

and for each $h \in \widetilde{L^0}$

$$|Sh| \leqslant q\tilde{h}.$$

In particular, for each B.f.s. X there holds $\|S\|_{\widetilde{X} \to \widetilde{X}} \leqslant q$.

Remark

Using Hahn-Banach-Kantorovitch theorem one may take $q = 1$.

Looking for S

Proposition

Let $g \in \widetilde{L^0} = \{f \in L^0 : \tilde{f} \in L^0\}$. Then for each $q > 1$ there is a linear operator S defined on $\widetilde{L^0}$ such that

$$Sg = \widetilde{g}$$

and for each $h \in \widetilde{L^0}$

$$|Sh| \leqslant q\tilde{h}.$$

In particular, for each B.f.s. X there holds $\|S\|_{\widetilde{X} \to \widetilde{X}} \leqslant q$.

Remark

Using Hahn-Banach-Kantorovich theorem one may take $q = 1$.

Looking for T

- For $a = (a_1, ..., a_n) \in \mathbb{R}^n$, a^* is the vector produced by permuting entries of $|a|$ in nonincreasing order.
- Writing $b \prec a$, for $a, b \in \mathbb{R}^n$ we understand that

$$\sum_{i=1}^{k} b_i^* \leqslant \sum_{i=1}^{k} a_i^* \text{ for each } 0 < k \leqslant n,$$

- A positive matrix $A = (a_{ij})_{i,j=1}^{n}$ (here positivity means that $0 \leqslant a_{ij}$ for all i, j, or equivalently $0 \leqslant Aa$ for each $0 \leqslant a \in \mathbb{R}^n$) is called **substochastic** when

$$\sum_{j=1}^{n} a_{ij} \leqslant 1 \text{ and } \sum_{j=1}^{n} a_{ji} \leqslant 1 \text{ for each } 0 < i \leqslant n. \tag{6}$$

- A positive matrix A is substochastic if and only if $Aa \prec a$ for each $0 \leqslant a \in \mathbb{R}^n$.

Looking for T

- For $a = (a_1, ..., a_n) \in \mathbb{R}^n$, a^* is the vector produced by permuting entries of $|a|$ in nonincreasing order.
- Writing $b \prec a$, for $a, b \in \mathbb{R}^n$ we understand that

$$\sum_{i=1}^{k} b_i^* \leqslant \sum_{i=1}^{k} a_i^* \text{ for each } 0 < k \leqslant n,$$

- A positive matrix $A = (a_{ij})_{i,j=1}^n$ (here positivity means that $0 \leqslant a_{ij}$ for all i, j, or equivalently $0 \leqslant Aa$ for each $0 \leqslant a \in \mathbb{R}^n$) is called **substochastic** when

$$\sum_{j=1}^{n} a_{ij} \leqslant 1 \text{ and } \sum_{j=1}^{n} a_{ji} \leqslant 1 \text{ for each } 0 < i \leqslant n. \tag{6}$$

- A positive matrix A is substochastic if and only if $Aa \prec a$ for each $0 \leqslant a \in \mathbb{R}^n$.

Looking for T

- For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, a^* is the vector produced by permuting entries of $|a|$ in nonincreasing order.
- Writing $b \prec a$, for $a, b \in \mathbb{R}^n$ we understand that

$$\sum_{i=1}^{k} b_i^* \leqslant \sum_{i=1}^{k} a_i^* \text{ for each } 0 < k \leqslant n,$$

- A positive matrix $A = (a_{ij})_{i,j=1}^n$ (here positivity means that $0 \leqslant a_{ij}$ for all i, j, or equivalently $0 \leqslant Aa$ for each $0 \leqslant a \in \mathbb{R}^n$) is called **substochastic** when

$$\sum_{j=1}^{n} a_{ij} \leqslant 1 \text{ and } \sum_{j=1}^{n} a_{ji} \leqslant 1 \text{ for each } 0 < i \leqslant n. \tag{6}$$

- A positive matrix A is substochastic if and only if $Aa \prec a$ for each $0 \leqslant a \in \mathbb{R}^n$.

Looking for T

- For $a = (a_1, ..., a_n) \in \mathbb{R}^n$, a^* is the vector produced by permuting entries of $|a|$ in nonincreasing order.
- Writing $b \prec a$, for $a, b \in \mathbb{R}^n$ we understand that

$$\sum_{i=1}^{k} b_i^* \leqslant \sum_{i=1}^{k} a_i^* \text{ for each } 0 < k \leqslant n,$$

- A positive matrix $A = (a_{ij})_{i,j=1}^n$ (here positivity means that $0 \leqslant a_{ij}$ for all i, j, or equivalently $0 \leqslant Aa$ for each $0 \leqslant a \in \mathbb{R}^n$) is called **substochastic** when

$$\sum_{j=1}^{n} a_{ij} \leqslant 1 \text{ and } \sum_{j=1}^{n} a_{ji} \leqslant 1 \text{ for each } 0 < i \leqslant n. \tag{6}$$

- A positive matrix A is substochastic if and only if $Aa \prec a$ for each $0 \leqslant a \in \mathbb{R}^n$.

Looking for T

- For $f, g \in L^1 + L^\infty$ we write $f \prec g$ when
$$\int_0^x f^*(t)dt \leqslant \int_0^x g^*(t)dt \text{ for each } x > 0.$$
- $K(x, f; L^1, L^\infty) = \int_0^x f^*(t)dt$
- A linear positive operator (in the sense that $0 \leqslant f$ implies $0 \leqslant Tf$) defined on $L^1 + L^\infty$, mapping continuously L^1 into L^1 and L^∞ into L^∞ with both norms less or equal one is called **substochastic**.
- A positive T is substochastic if and only if $Tf \prec f$ for each $f \in L^1 + L^\infty$.

Looking for T

- For $f, g \in L^1 + L^\infty$ we write $f \prec g$ when

$$\int_0^x f^*(t)dt \leqslant \int_0^x g^*(t)dt \text{ for each } x > 0.$$

- $K(x, f; L^1, L^\infty) = \int_0^x f^*(t)dt$
- A linear positive operator (in the sense that $0 \leqslant f$ implies $0 \leqslant Tf$) defined on $L^1 + L^\infty$, mapping continuously L^1 into L^1 and L^∞ into L^∞ with both norms less or equal one is called **substochastic**.
- A positive T is substochastic if and only if $Tf \prec f$ for each $f \in L^1 + L^\infty$.

Looking for T

- For $f, g \in L^1 + L^\infty$ we write $f \prec g$ when

$$\int_0^x f^*(t)dt \leqslant \int_0^x g^*(t)dt \text{ for each } x > 0.$$

- $K(x, f; L^1, L^\infty) = \int_0^x f^*(t)dt$
- A linear positive operator (in the sense that $0 \leqslant f$ implies $0 \leqslant Tf$) defined on $L^1 + L^\infty$, mapping continuously L^1 into L^1 and L^∞ into L^∞ with both norms less or equal one is called **substochastic**.
- A positive T is substochastic if and only if $Tf \prec f$ for each $f \in L^1 + L^\infty$.

Looking for T

- For $f, g \in L^1 + L^\infty$ we write $f \prec g$ when

$$\int_0^x f^*(t)dt \leqslant \int_0^x g^*(t)dt \text{ for each } x > 0.$$

- $K(x, f; L^1, L^\infty) = \int_0^x f^*(t)dt$
- A linear positive operator (in the sense that $0 \leqslant f$ implies $0 \leqslant Tf$) defined on $L^1 + L^\infty$, mapping continuously L^1 into L^1 and L^∞ into L^∞ with both norms less or equal one is called **substochastic**.
- A positive T is substochastic if and only if $Tf \prec f$ for each $f \in L^1 + L^\infty$.

Looking for T

Theorem (Hardy-Littlewood-Pólya)

Let $0 \leqslant a, b \in \mathbb{R}^n$. If $b \prec a$ then there exists a substochastic matrix A such that $Aa = b$.

Theorem (Calderón)

Let $0 \leqslant f, g \in L^1 + L^\infty$ and suppose that $g \prec f$. Then there is a substochastic operator T such that $Tf = g$.

Looking for T

Theorem (Hardy-Littlewood-Pólya)

Let $0 \leqslant a, b \in \mathbb{R}^n$. If $b \prec a$ then there exists a substochastic matrix A such that $Aa = b$.

Theorem (Calderón)

Let $0 \leqslant f, g \in L^1 + L^\infty$ and suppose that $g \prec f$. Then there is a substochastic operator T such that $Tf = g$.

Looking for T

Proposition

Let X be a Banach function space and $f \in \widetilde{X} + L^\infty$. Then

$$K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{g}; X, L^\infty).$$

Therefore $K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty)$ means that

$$\tilde{f} \prec \tilde{g}.$$

By Calderón theorem there is a substochastic operator T such that

$$T\tilde{g} = \tilde{f}.$$

But

$$T : (\widetilde{L^1}, L^\infty) \not\to (\widetilde{L^1}, L^\infty)!$$

Looking for T

Proposition

Let X be a Banach function space and $f \in \widetilde{X} + L^\infty$. Then

$$K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{g}; X, L^\infty).$$

Therefore $K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty)$ means that

$$\tilde{f} \prec \tilde{g}.$$

By Calderón theorem there is a substochastic operator T such that

$$T\tilde{g} = \tilde{f}.$$

But

$$T : (\widetilde{L^1}, L^\infty) \not\to (\widetilde{L^1}, L^\infty)!$$

Looking for T

Proposition

Let X be a Banach function space and $f \in \widetilde{X} + L^\infty$. Then

$$K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{g}; X, L^\infty).$$

Therefore $K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty)$ means that

$$\tilde{f} \prec \tilde{g}.$$

By Calderón theorem there is a substochastic operator T such that

$$T\tilde{g} = \tilde{f}.$$

But

$$T : (\widetilde{L^1}, L^\infty) \not\to (\widetilde{L^1}, L^\infty)!$$

Looking for T

Proposition

Let X be a Banach function space and $f \in \widetilde{X} + L^\infty$. Then

$$K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{g}; X, L^\infty).$$

Therefore $K(\cdot, f; \widetilde{L^1}, L^\infty) \leqslant K(\cdot, g; \widetilde{L^1}, L^\infty)$ means that

$$\tilde{f} \prec \tilde{g}.$$

By Calderón theorem there is a substochastic operator T such that

$$T\tilde{g} = \tilde{f}.$$

But

$$T : (\widetilde{L^1}, L^\infty) \not\to (\widetilde{L^1}, L^\infty)!$$

Looking for T

Definition

X, Y - Banach function spaces. An operator $T : X \mapsto Y$ is **monotone** if it is positive (i.e. $0 \leqslant f$ implies $0 \leqslant Tf$) and for each nonincreasing $0 \leqslant f \in X$, Tf is also nonincreasing.

Proposition

If a bounded operator $T : X \to Y$ is monotone, then $T : \widetilde{X} \to \widetilde{Y}$ with $\|T\|_{\widetilde{X} \to \widetilde{Y}} \leqslant \|T\|_{X \to Y}$.

Looking for T

Definition

X, Y - Banach function spaces. An operator $T : X \mapsto Y$ is **monotone** if it is positive (i.e. $0 \leqslant f$ implies $0 \leqslant Tf$) and for each nonincreasing $0 \leqslant f \in X$, Tf is also nonincreasing.

Proposition

If a bounded operator $T : X \to Y$ is monotone, then $T : \widetilde{X} \to \widetilde{Y}$ with $\|T\|_{\widetilde{X} \to \widetilde{Y}} \leqslant \|T\|_{X \to Y}$.

Looking for T

Theorem (Bennett-Sharpley 1986, KL 2015)
Let $0 \leqslant a, b \in \mathbb{R}^n$ be both nonincreasing. If $b \prec a$ then there exists a substochastic monotone matrix A such that $Aa = b$.

Theorem (Bennett-Sharpley 1986, KL 2015)
Let $0 \leqslant f, g \in L^1 + L^\infty$ be both nonincreasing and suppose that $g \prec f$. Then there is a substochastic monotone operator T such that $Tf = g$.

Looking for T

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant a, b \in \mathbb{R}^n$ be both nonincreasing. If $b \prec a$ then there exists a substochastic monotone matrix A such that $Aa = b$.

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant f, g \in L^1 + L^\infty$ be both nonincreasing and suppose that $g \prec f$. Then there is a substochastic monotone operator T such that $Tf = g$.

Looking for T

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant a, b \in \mathbb{R}^n$ be both nonincreasing. If $b \prec a$ then there exists a substochastic monotone matrix A such that $Aa = b$.

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant f, g \in L^1 + L^\infty$ be both nonincreasing and suppose that $g \prec f$. Then there is a substochastic monotone operator T such that $Tf = g$.

Looking for T

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant a, b \in \mathbb{R}^n$ be both nonincreasing. If $b \prec a$ then there exists a substochastic monotone matrix A such that $Aa = b$.

Theorem (Bennett-Sharpley 1986, KL 2015)

Let $0 \leqslant f, g \in L^1 + L^\infty$ be both nonincreasing and suppose that $g \prec f$. Then there is a substochastic monotone operator T such that $Tf = g$.

Theorem (Avni-Cwikel 2013)

If (X_0, X_1) is a Calderón couple then also (X_0^p, X_1^p) is a Calderón couple for $1 < p < \infty$.

Corollary

The couple $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.

Problem

Suppose (X_0, X_1) is a Calderón couple, where X_0, X_1 are B.f.s. and let $0 \leqslant f, g \in X_0 + X_1$ be both nonincreasing with

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1).$$

Then there exists a positive $T : (X_0, X_1) \to (X_0, X_1)$ satisfying $Tg = f$. May we choose T to be monotone?

Theorem (Avni-Cwikel 2013)

If (X_0, X_1) is a Calderón couple then also (X_0^p, X_1^p) is a Calderón couple for $1 < p < \infty$.

Corollary

The couple $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.

Problem

Suppose (X_0, X_1) is a Calderón couple, where X_0, X_1 are B.f.s. and let $0 \leqslant f, g \in X_0 + X_1$ be both nonincreasing with

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1).$$

Then there exists a positive $T : (X_0, X_1) \to (X_0, X_1)$ satisfying $Tg = f$. May we choose T to be monotone?

Theorem (Avni-Cwikel 2013)

If (X_0, X_1) is a Calderón couple then also (X_0^p, X_1^p) is a Calderón couple for $1 < p < \infty$.

Corollary

The couple $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.

Problem

Suppose (X_0, X_1) is a Calderón couple, where X_0, X_1 are B.f.s. and let $0 \leqslant f, g \in X_0 + X_1$ be both nonincreasing with

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1).$$

Then there exists a positive $T : (X_0, X_1) \to (X_0, X_1)$ satisfying $Tg = f$. May we choose T to be monotone?

Theorem (Avni-Cwikel 2013)

If (X_0, X_1) is a Calderón couple then also (X_0^p, X_1^p) is a Calderón couple for $1 < p < \infty$.

Corollary

The couple $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.

Problem

Suppose (X_0, X_1) is a Calderón couple, where X_0, X_1 are B.f.s. and let $0 \leqslant f, g \in X_0 + X_1$ be both nonincreasing with

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1).$$

Then there exists a positive $T : (X_0, X_1) \to (X_0, X_1)$ satisfying $Tg = f$. May we choose T to be monotone?

Theorem (Avni-Cwikel 2013)

If (X_0, X_1) is a Calderón couple then also (X_0^p, X_1^p) is a Calderón couple for $1 < p < \infty$.

Corollary

The couple $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.

Problem

Suppose (X_0, X_1) is a Calderón couple, where X_0, X_1 are B.f.s. and let $0 \leqslant f, g \in X_0 + X_1$ be both nonincreasing with

$$K(\cdot, f; X_0, X_1) \leqslant K(\cdot, g; X_0, X_1).$$

Then there exists a positive $T : (X_0, X_1) \to (X_0, X_1)$ satisfying $Tg = f$. May we choose T to be monotone?

Theorem (Maligranda-KL 2015)

Let X, X_0, X_1 be B.f.s. with the Fatou property and such that C is bounded on them.

(a) If the dilation operator σ_τ is boundedon X_0 and X_1 for some $0 < \tau < 1$, then

$$\varphi(CX_0, CX_1) = C[\varphi(X_0, X_1)]. \tag{7}$$

(b) If X is a symmetric space such that C is bounded on $\varphi(L^1, X)$, then

$$\varphi(L^1, CX) = C[\varphi(L^1, X)]. \tag{8}$$

(c) If either X_0, X_1 are symmetric spaces, or C^* is bounded on both X_0 and X_1, then

$$\varphi(\widetilde{X_0}, \widetilde{X_1}) = [\varphi(X_0, X_1)]^{\sim}. \tag{9}$$

Theorem (Maligranda-KL 2015)

Let X, X_0, X_1 be B.f.s. with the Fatou property and such that C is bounded on them.

(a) If the dilation operator σ_τ is boundedon X_0 and X_1 for some $0 < \tau < 1$, then

$$\varphi(CX_0, CX_1) = C[\varphi(X_0, X_1)]. \tag{7}$$

(b) If X is a symmetric space such that C is bounded on $\varphi(L^1, X)$, then

$$\varphi(L^1, CX) = C[\varphi(L^1, X)]. \tag{8}$$

(c) If either X_0, X_1 are symmetric spaces, or C^* is bounded on both X_0 and X_1, then

$$\varphi(\widetilde{X_0}, \widetilde{X_1}) = [\varphi(X_0, X_1)]^{\sim}. \tag{9}$$

Theorem (Maligranda-KL 2015)

Let X, X_0, X_1 be B.f.s. with the Fatou property and such that C is bounded on them.

(a) If the dilation operator σ_τ is boundedon X_0 and X_1 for some $0 < \tau < 1$, then

$$\varphi(CX_0, CX_1) = C[\varphi(X_0, X_1)]. \tag{7}$$

(b) If X is a symmetric space such that C is bounded on $\varphi(L^1, X)$, then

$$\varphi(L^1, CX) = C[\varphi(L^1, X)]. \tag{8}$$

(c) If either X_0, X_1 are symmetric spaces, or C^* is bounded on both X_0 and X_1, then

$$\varphi(\widetilde{X_0}, \widetilde{X_1}) = [\varphi(X_0, X_1)]^{\sim}. \tag{9}$$

Theorem (Maligranda-KL 2015)

Let $0 < \theta < 1$. Assume that X, X_0, X_1 are complex Banach function spaces with the Fatou property, $L^1 \cap L^\infty \hookrightarrow X$, and such that the operator $f \mapsto C|f|$ is bounded on all of them.

(a) If the dilation operator σ_a for some $0 < a < 1$ is bounded on X_0 and X_1 and at least one of the spaces X_0 or X_1 is order continuous, then

$$[CX_0, CX_1]_\theta = C([X_0, X_1]_\theta).$$

(b) If X is a symmetric space, then

$$[L^1, CX]_\theta = C([L^1, X]_\theta).$$

(c) If at least one of the spaces X_0, X_1 is order continuous and either X_0 and X_1 are symmetric spaces or C^* is bounded on X_0 and X_1, then

$$[\widetilde{X_0}, \widetilde{X_1}]_\theta = ([X_0, X_1]_\theta)^\sim.$$

Theorem (Maligranda-KL 2015)

Let $0 < \theta < 1$. Assume that X, X_0, X_1 are complex Banach function spaces with the Fatou property, $L^1 \cap L^\infty \hookrightarrow X$, and such that the operator $f \mapsto C|f|$ is bounded on all of them.

(a) If the dilation operator σ_a for some $0 < a < 1$ is bounded on X_0 and X_1 and at least one of the spaces X_0 or X_1 is order continuous, then

$$[CX_0, CX_1]_\theta = C([X_0, X_1]_\theta).$$

(b) If X is a symmetric space, then

$$[L^1, CX]_\theta = C([L^1, X]_\theta).$$

(c) If at least one of the spaces X_0, X_1 is order continuous and either X_0 and X_1 are symmetric spaces or C^* is bounded on X_0 and X_1, then

$$[\widetilde{X_0}, \widetilde{X_1}]_\theta = ([X_0, X_1]_\theta)^{\sim}.$$

Theorem (Maligranda-KL 2015)

Let $0 < \theta < 1$. Assume that X, X_0, X_1 are complex Banach function spaces with the Fatou property, $L^1 \cap L^\infty \hookrightarrow X$, and such that the operator $f \mapsto C|f|$ is bounded on all of them.

(a) If the dilation operator σ_a for some $0 < a < 1$ is bounded on X_0 and X_1 and at least one of the spaces X_0 or X_1 is order continuous, then

$$[CX_0, CX_1]_\theta = C([X_0, X_1]_\theta).$$

(b) If X is a symmetric space, then

$$[L^1, CX]_\theta = C([L^1, X]_\theta).$$

(c) If at least one of the spaces X_0, X_1 is order continuous and either X_0 and X_1 are symmetric spaces or C^* is bounded on X_0 and X_1, then

$$[\widetilde{X_0}, \widetilde{X_1}]_\theta = ([X_0, X_1]_\theta)^{\sim}.$$

Theorem (Maligranda-KL 2015)

Let X_0, X_1 be B.f.s. with the Fatou property. If C and C^* are bounded on X_i for $i = 0, 1$ and F is an interpolation functor with the homogenity property, that is, $F(X_0(w), X_1(w)) = F(X_0, X_1)(w)$ for any weight w, then

$$F(CX_0, CX_1) = CF(X_0, X_1). \tag{10}$$

In particular,

$$(CX_0, CX_1)^K_G = C[(X_0, X_1)^K_G]. \tag{11}$$

Thank you!