Optimal approximation of multivariate periodic Sobolev functions

Thomas Kühn

Universität Leipzig, Germany

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Topic of the talk

- Approximation of d-variate periodic functions in
  - isotropic Sobolev spaces $H^s(\mathbb{T}^d)$
  - Sobolev spaces of dominating mixed smoothness $H^s_{mix}(\mathbb{T}^d)$
  - more general periodic spaces on the d-dimensional torus $\mathbb{T}^d$
- The error is measured in the L_2-norm and in the sup-norm.
- Special emphasis on hidden constants, especially their dependence on the smoothness parameter $s > 0$ and the dimension $d \in \mathbb{N}$.

The talk is based on results from the following papers:

- T. Kühn, W. Sickel and T. Ullrich,
  Approximation numbers of Sobolev embeddings – Sharp constants and tractability, J. Complexity 30 (2014), 95–116.
- T. Kühn, W. Sickel and T. Ullrich,
  Approximation of mixed order Sobolev functions on the d-torus – Asymptotics, preasymptotics and d-dependence,
  Constr. Approx. (Online First 2015), arXiv:1312.6386
- F. Cobos, T. Kühn and W. Sickel,
  Optimal approximation of multivariate periodic Sobolev functions in the sup-norm, submitted 2014, arXiv:1505.02636
- S. Mayer, T. Kühn and T. Ullrich,
  Counting via entropy: new preasymptotics for the approximation numbers of Sobolev embeddings, submitted 2015, arXiv:1505.00631

Approximation numbers

- **Approximation numbers** (also called linear widths) of bounded linear operators $T : X \to Y$ in Banach spaces

$$a_n(T : X \to Y) := \inf\{\|T - A\| : \operatorname{rank} A < n\}$$

- **Many nice properties**

(1)	Monotonicity	$\|T\| = a_1(T) \geq a_2(T) \geq \ldots \geq 0$
(2)	Additivity	$a_{n+k-1}(S + T) \leq a_n(S) + a_k(T)$
(3)	Multiplicativity	$a_{n+k-1}(S \circ T) \leq a_n(S) \cdot a_k(T)$
(4)	Rank property	$\operatorname{rank} T < n \Longrightarrow a_n(T) = 0$
(5)	Norming property	$a_n(id : \ell_2^n \to \ell_2^n) = 1$

Relation to singular numbers

- **Singular numbers** (= singular values, known from SVD) of compact linear operators $T : H \to F$ between two Hilbert spaces

$$s_n(T) := \sqrt{\lambda_n(T^*T)}$$

- **Schmidt representation** of compact operators $T : H \to F$
$\exists$ orthonormal systems $(e_n) \subset H$ and $(f_n) \subset F$ s.t.

$$Th = \sum_n s_n(T)\langle h, e_n\rangle f_n \quad \text{for all } h \in H\,.$$

- **Characterization by best approximations**

$$s_n(T) = \inf\{\|T - A\| : \operatorname{rank} A < n\} = a_n(T)$$

- For operators on Hilbert spaces, approximation numbers coincide with all other s-numbers (like Kolmogorov, Gelfand, Weyl numbers, ...)

Interpretation in terms of algorithms

- Every operator $A : X \to Y$ of finite rank n can be written as

$$Ax = \sum_{j=1}^{n} L_j(x)\, y_j \qquad \text{for all } x \in X$$

with linear functionals $L_j \in X^*$ and vectors $y_j \in Y$.
$\curvearrowright$ A is a linear algorithm using n arbitrary linear informations

- worst-case error of the algorithm A

$$err^{wor}(A) := \sup_{\|x\| \le 1} \| Tx - Ax \| = \| T - A \|$$

- n-th minimal worst-case error of the approximation problem for T (w.r.t. linear algorithms and arbitrary linear information)

$$err_n^{wor}(T) := \inf_{\operatorname{rank} A \le n} err^{wor}(A) = a_{n+1}(T)$$

Sobolev embeddings

- **Well-known:**
  - For isotropic spaces on the d-dimensional torus $\mathbb{T}^d$

$$c_{s,d} \cdot n^{-s/d} \leq a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq C_{s,d} \cdot n^{-s/d}$$

  - For spaces of dominating mixed smoothness

$$c_{s,d} \cdot \left[\frac{(\log n)^{d-1}}{n}\right]^s \leq a_n(I_d : H^s_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq C_{s,d} \cdot \left[\frac{(\log n)^{d-1}}{n}\right]^s$$

- **Almost nothing known:**

  How do the constants $c_{s,d}$ and $C_{s,d}$ depend on s and d ???

  This is essential for high-dimensional numerical problems, and also for tractability questions in information-based complexity!

Some remarks

- Of course, the constants heavily depend on the chosen norms.

  $\curvearrowright$ First we have to fix (somehow natural) norms.

  For all our norms, we will have norm one embeddings into $L_2(\mathbb{T}^d)$.

- For example, for smoothness $s = 1$, the asymptotic rates are

$$\alpha_n := n^{-1/d} \quad \text{and} \quad \beta_n := \frac{(\log n)^{d-1}}{n}.$$

  In high dimensions, one has to wait exponentially long until these rates become visible, as one can see from the following examples.

- Isotropic case.

  $n = 10^d$ (very large) $\curvearrowright$ $\alpha_n = \frac{1}{10}$ (poor error estimate)

- Mixed case. (Dimension $d + 1$)

  Even worse, $n = d^d$ $\curvearrowright$ $\beta_n = (\log d)^d \gg 1$ (trivial estimate)

- $\curvearrowright$ We need information on the constants,

  and preasymptotic estimates (for small n)

Natural norms

- $H^m(\mathbb{T}^d)$, with integer smoothness $m \in \mathbb{N}$
- **Classical norm** (all partial derivatives)

$$\| f \,|H^m(\mathbb{T}^d)\| := \Big(\sum_{|\alpha| \le m} \| D^\alpha f \,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2}$$

- **Modified classical norm** (only highest derivatives in each coordinate)

$$\| f \,|H^m(\mathbb{T}^d)\|^* := \Big(\| f \,|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^{d} \Big\| \frac{\partial^m f}{\partial x_j^m} \Big| L_2(\mathbb{T}^d) \Big\|^2 \Big)^{1/2}$$

Norms via Fourier coefficients

- These norms can be rewritten in terms of Fourier coefficients of f,

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ikx} dx \quad , \quad k \in \mathbb{Z}^d$$

via Parseval's identity and $c_k(D^\alpha f) = (ik)^\alpha c_k(f)$.

- For the natural norm one has **equivalence**

$$\| f \,|H^m(\mathbb{T}^d)\| \sim \left(\sum_{k \in \mathbb{Z}^d} \left(1 + \sum_{j=1}^d |k_j|^2\right)^m |c_k(f)|^2 \right)^{1/2}$$

with equivalence constants independent on d.

- For the modified natural norm one has even **equality**

$$\| f \,|H^m(\mathbb{T}^d)\|^* = \left(\sum_{k \in \mathbb{Z}^d} \left(1 + \sum_{j=1}^d |k_j|^{2m}\right) |c_k(f)|^2 \right)^{1/2} .$$

Norms for fractional smoothness $s > 0$

- Let $s > 0$, $d \in \mathbb{N}$ and $0 < p < \infty$.

  $H^{s,p}(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

  $$\|f|H^{s,p}(\mathbb{T}^d)\| := \Big(\sum_{k \in \mathbb{Z}^d} w_{s,p}(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty ,$$

  where the weights are $$w_{s,p}(k) := \Big(1 + \sum_{j=1}^{d} |k_j|^p\Big)^{s/p} .$$

- For fixed $s > 0$ and $d \in \mathbb{N}$, all these norms are equivalent.
  Clearly, the equivalence constants depend on d.
  But all spaces $H^{s,p}(\mathbb{T}^d)$, $0 < p < \infty$, coincide as vector spaces.

- Very useful is the semigroup property of these weights,

  $$w_{s,p}(k) = w_{1,p}(k)^s$$

  which allows reduction to the case of smoothness $s = 1$.

Relation to the classical norms

- For the natural norm we have **equivalence**

$$\| f \,|H^m(\mathbb{T}^d)\| \sim \| f \,|H^{m,2}(\mathbb{T}^d)\|$$

  with equivalence constants independent on d.

- For the modified natural norm one has even **equality**

$$\| f \,|H^m(\mathbb{T}^d)\|^* = \| f \,|H^{m,2m}(\mathbb{T}^d)\|$$

Norms on $H^s_{mix}(\mathbb{T}^d)$

- Let $s > 0$, $d \in \mathbb{N}$ and $0 < p < \infty$.

  $H^{s,p}_{mix}(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

  $$\|f|H^{s,p}(\mathbb{T}^d)\| := \Big(\sum_{k \in \mathbb{Z}^d} w^{mix}_{s,p}(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty \,,$$

  where the weights are now $\qquad w^{mix}_{s,p}(k) := \prod_{j=1}^{d} (1 + |k_j|^p)^{s/p} \,.$

- Again, for fixed $s > 0$ and $d \in \mathbb{N}$, all these norms are equivalent. Clearly, the equivalence constants depend on d. But all spaces $H^{s,p}_{mix}(\mathbb{T}^d)$, $0 < p < \infty$, coincide as vector spaces.

More general periodic spaces

- Given any weights $w(k) \geq 1$, $k \in \mathbb{Z}^d$, we define $F_d(w)$ as the space of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|F_d(w)\| := \Big(\sum_{k\in\mathbb{Z}^d} w(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty\,.$$

- Examples: all Sobolev spaces $H^{s,p}(\mathbb{T}^d)$ and $H^{s,p}_{mix}(\mathbb{T}^d)$
- We have compact embeddings

$$F_d(w) \hookrightarrow L_2(\mathbb{T}^d) \quad \Longleftrightarrow \quad \lim_{|k|\to\infty} 1/w(k) = 0$$

$$F_d(w) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \Longleftrightarrow \quad \sum_{k\in\mathbb{Z}^d} 1/w(k)^2 < \infty\,.$$

Reduction to sequence spaces

Commutative diagram

$$\begin{array}{ccc} F_d(w) & \xrightarrow{\quad I_d \quad} & L_2(\mathbb{T}^d) \\ \Big\downarrow A & & \Big\uparrow B \\ \ell_2(\mathbb{Z}^d) & \xrightarrow{\quad D \quad} & \ell_2(\mathbb{Z}^d) \end{array}$$

$$Af := (w(k)\, c_k(f))_{k\in\mathbb{Z}^d} \quad , \quad B\xi := \sum_{k\in\mathbb{Z}^d} \xi_k\, e^{ikx} \quad , \quad D(\xi_k) := (\xi_k / w(k))$$

A and B unitary operators $\curvearrowright$ $a_n(I_d) = a_n(D) = s_n(D) = \sigma_n$

where $(\sigma_n)_{n\in\mathbb{N}}$ is the non-increasing rearrangement of $(1/w(k))_{k\in\mathbb{Z}^d}$.

Isotropic Sobolev spaces

- If $(\sigma_n)_{n\in\mathbb{N}}$ is the non-increasing rearrangement of $\left(\frac{1}{w_{s,p}(k)}\right)_{k\in\mathbb{Z}^d}$, then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \sigma_n\,.$$

- The "sequence" $(w_{s,p}(k))_{k\in\mathbb{Z}^d}$ is piecewise constant, it attains all values $(1+r^p)^{s/p}$, $r \in \mathbb{N}$, each of them at least $2d$ times, for $k = \pm re_1, \pm re_2 \ldots, \pm re_d$.
- For $r, d \in \mathbb{N}$ define $N(r,d) := card\{k \in \mathbb{Z}^d : \sum_{j=1}^d |k_j|^p \le r^p\}$.

Lemma

If $N(r-1,d) < n \le N(r,d)$, then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (1+r^p)^{-s/p}\,.$$

- In principle, this gives $a_n(I_d)$ for all n, but the exact computation of the cardinalities $N(r,d)$ is impossible. The hard work is to find good estimates, using combinatorial and volume arguments.

Asymptotic constants, $n \to \infty$

- Let B_p^d denote the unit ball in $(\mathbb{R}^d, \|.\|_p)$. Using volume estimates, we can show the existence of asymptotically optimal constants.

Theorem (KSU 2014)

Let $0 < s, p < \infty$ *and* $d \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} n^{s/d} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = vol(B_p^d)^{s/d} \sim d^{-s/p}$$

- The asymptotic constant is of order
  $d^{-s/2}$ for the natural norm $(p = 2)$,
  $d^{-1/2}$ for the modified natural norm $(p = 2s)$.
  (In the paper only for $p = 1, 2, 2s$, but proof works for arbitrary p.)
- We get the correct order $n^{-s/d}$ of the a_n in n and the
  exact decay rate $d^{-s/p}$ of the constants in d.
- Polynomial decay in d of the constants helps in error estimates!

Estimates for large n

Theorem (KSU 2014, case $p = 1$)

Let $s > 0$ and $n \geq 6^d/3$. Then

$$d^{-s} n^{-s/d} \leq a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq (4e)^s d^{-s} n^{-s/d} .$$

- We have similar estimates for all other $0 < p < \infty$, but for $p = 1$ the constants are nicer.
- Note the correct d-dependence d^{-s} of the constants!
- Proof: via combinatorial estimates of the cardinalities $N(r, d)$

Preasymptotic estimates – small n

Theorem (KSU 2014)

Let $p = 1$ and $2 \le n \le 2^d$. Then

$$\left(\frac{1}{2 + \log_2 n}\right)^s \le a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{\log_2(2d+1)}{\log_2 n}\right)^s .$$

- This estimate was shown by combinatorial arguments, which only work for $p = 1$. Using a relation to entropy numbers, we could close the gap between lower and upper bounds and treat arbitrary p's.

Theorem (KMU 2015)

Let $s > 0$, $0 < p < \infty$ and $2 \le n \le 2^d$. Then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim \left(\frac{\log_2(1 + d/\log_2 n)}{\log_2 n}\right)^{s/p} .$$

(We have explicit expressions for the hidden constants.)

Dominating mixed derivatives

- The strategy for isotropic spaces can be used analogously in the dominating mixed smoothness case.
  Of course, the combinatorial estimates are different, and harder!

Theorem (KSU 2015 - asymptotic constants)

Let $s > 0$ and $d \in \mathbb{N}$. Then, for all $0 < p < \infty$, it holds

$$\lim_{n\to\infty} \frac{n^s a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \left[\frac{2^d}{(d-1)!}\right]^s$$

- In the paper this was shown only for $p = 1, 2, 2s$,
  but the proof works for all p.
- Interesting fact: For all $0 < p < \infty$ the limit is the same.
- The asymptotic constant decays super-exponentially in d.

Estimates for large n

- As examples, we give some estimates for the norms with $p = 1$.

Theorem (KSU 2015)

Let $s > 0$ and $d \in \mathbb{N}$. Then, for $n \geq 27^d$, it holds

$$a_n(I_d : H^{s,1}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left[\frac{3^d}{(d-1)!}\right]^s \frac{(\log n)^{s(d-1)}}{n^s}$$

For $n > (12e^2)^d$ we have, with $c = \frac{2}{2+\log 12}$,

$$a_n(I_d : H^{s,1}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \left[\frac{3\,c^d}{d!}\right]^s \frac{(\log n)^{s(d-1)}}{n^s}$$

- For the constant in the upper estimate we still have super-exponential decay in d.
  (The only difference to the limit is 3^d instead of 2^d.)

Preasymptotic estimates, small n

Theorem (KSU 2015)

Let $s > 0$ and $d \in \mathbb{N}$, $d \geq 2$. Then, for $9 \leq n \leq d\, 2^{2d-1}$, it holds

$$a_n(I_d : H^{s,1}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{e^2}{n}\right)^{\frac{s}{2+\log_2 d}}$$

- Note that the bound is non-trivial for all n in the given range, since $e^2 < 9$.
- We have also similar (non-matching) lower estimates.
  But they show, that one has to wait exponentially long until one can "see" the correct asymptotic rate n^{-s}, ignoring the log-terms.

Approximation in the sup-norm

- It is well-known that

$$H^s(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \iff \quad s > \frac{d}{2}$$

$$H^s_{mix}(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \iff \quad s > \frac{1}{2}$$

- The asymptotic behaviour of the approximation numbers is also well-known, up to multiplicative constants,

$$a_n(I_d : H^s(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \sim n^{1/2 - s/d}$$

$$a_n(I_d : H^s_{mix}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \sim n^{1/2 - s}(\log n)^{s(d-1)}$$

- **Problem.** Find estimates for the hidden constants and the families of norms, with parameters $0 < p < \infty$.

From L_2-approximation to L_∞-approximation

- General spaces $F_d(w)$: all $f \in L_2(\mathbb{T}^d)$ with $\sum_k w(k)^2 |c_k(f)|^2 < \infty$

  Wiener algebra $\mathcal{A}(\mathbb{T}^d)$: all $f \in L_2(\mathbb{T}^d)$ with $\sum_k |c_k(f)| < \infty$

- $F_d(w) \hookrightarrow \mathcal{A}(\mathbb{T}^d)$ / $C(\mathbb{T}^d)$ / $L_\infty(\mathbb{T}^d) \iff \sum_{k \in \mathbb{Z}^d} w(k)^{-2} < \infty$

  In this case, the embeddings are even compact.

Theorem (CKS 2014)

Let $F_d(w) \hookrightarrow L_\infty(\mathbb{T}^d)$. Then

$$a_n(I_d : F_d(w) \to L_\infty(\mathbb{T}^d)) = \left(\sum_{j=n}^{\infty} a_j(I_d : F_d(w) \to L_2(\mathbb{T}^d))^2 \right)^{1/2}$$

The same holds for $\mathcal{A}(\mathbb{T}^d)$ and $C(\mathbb{T}^d)$ instead of $L_\infty(\mathbb{T}^d)$.

Sketch of proof

Upper estimate.

$$\begin{array}{ccccccc} F_d(w) & \xrightarrow{I_d} & \mathcal{A}(\mathbb{T}^d) & \longrightarrow & C(\mathbb{T}^d) & \longrightarrow & L_\infty(\mathbb{T}^d) \\ \Big\downarrow A & & \Big\uparrow B & & & & \\ \ell_2(\mathbb{Z}^d) & \xrightarrow{D} & \ell_1(\mathbb{Z}^d) & & & & \end{array}$$

$$Af := (w(k)\, c_k(f))_{k\in\mathbb{Z}^d} \quad , \quad B\xi := \sum_{k\in\mathbb{Z}^d} \xi_k\, e^{ikx} \quad , \quad D(\xi_k) := \left(\frac{\xi_k}{w(k)}\right)$$

$$\curvearrowright \quad a_n(I_d) \le \|A\| \cdot a_n(D) \cdot \|B\| = \left(\sum_{j=n}^\infty \sigma_n^2\right)^{1/2}$$

where $(\sigma_n)_{n\in\mathbb{N}}$ is the non-increasing rearrangement of $\left(\frac{1}{w(k)}\right)_{k\in\mathbb{Z}^d}$.

Remember: $\sigma_n = a_n(I_d : F_d(w) \to L_2(\mathbb{T}^d))$

Lower estimate: via absolutely 2-summing operators

An example - application to $H^{s,p}(\mathbb{T}^d)$

- The relation

$$\lim_{n\to\infty} n^{s/d} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \mathrm{vol}(B_p^d)^{s/d}$$

implies

Theorem (CKS 2014, asymptotic constants - isotropic spaces)

Let $d \in \mathbb{N}$, $s > d/2$ and $0 < p < \infty$. Then

$$\lim_{n\to\infty} n^{s/d-1/2} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) = \sqrt{\frac{d}{2s-d}} \cdot vol(B_p^d)^{s/d}$$

- Shift in the exponent of n by $\frac{1}{2}$, additional correction factor $\sqrt{\frac{d}{2s-d}}$.
- The same holds for the target space $C(\mathbb{T}^d)$, and also for the Wiener algebra $\mathcal{A}(\mathbb{T}^d)$.
- Similarly one can translate estimates of a_n for large n / small n.

Final remarks

- All this can be done also for Sobolev spaces of dominating mixed smoothness. For example, the relation

$$\lim_{n\to\infty} \frac{n^s a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \left[\frac{2^d}{(d-1)!}\right]^s$$

implies the following

Theorem (CKS 2014, asymptotic constants - mixed spaces)

Let $d \in \mathbb{N}$, $s > 1/2$ *and* $0 < p < \infty$. *Then*

$$\lim_{n\to\infty} \frac{n^{s-1/2} a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \frac{1}{\sqrt{2s-1}} \left[\frac{2^d}{(d-1)!}\right]^s$$

- Again: shift in the exponent by $\frac{1}{2}$ and additional correction factor.
- Open questions: Preasymptotic estimates? Tractability?

Thank you for your attention!