Sobolev spaces on metrizable groups

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Motivations

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- $\mathbb{R}^n$ - spaces $W^{k,p}$ and H^s
- Riemannian manifolds
- $\mathbb{T}^n$ - spaces defined by the Fourier transform
- Metric spaces - Hajłasz and Newtonian spaces
- Heisenberg groups and other special cases of locally compact groups

Outline

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1. Why locally compact abelian groups?
2. Sobolev spaces on locally compact abelian groups
3. What if we had a metric?

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Topological groups

Topological groups

Definition

- We say that $(G, \cdot, e, ^{-1})$ together with a topology $\tau \subset 2^G$ is a **topological group** if both maps

$$\begin{aligned} \cdot : G \times G &\longrightarrow G; & (a, b) &\longmapsto a \cdot b, \\ ^{-1} : G &\longrightarrow G; & a &\longmapsto a^{-1} \end{aligned}$$

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  are continuous.
- We say that a topological space is **locally compact** if every point has a neighbourhood which closure is compact.
- Locally compact group is a topological group which is a locally compact Hausdorff space.

Locally compact groups

Example

- $(\mathbb{R}^n, +, -, 0)$,

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- $(\mathbb{R} \setminus \{0\}, \cdot, ^{-1}, 1)$,
- $(\mathbb{Q}, +, -, 0)$ (with discrete topology),
- Lie groups,
- $(\mathbb{Q}_p, +, -, 0)$ (p is a prime number).

The Haar measure

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Theorem

Let G be a locally compact group. Then there exists exactly one non-zero, left-invariant Radon measure μ.

The Dual group

Definition

Let G be a locally compact abelian group. The set $\widehat{G}$ (or $G^\wedge$) of all continuous homomorphisms from G to $S^1 \subset \mathbb{C}$ is called a dual group of G. We equip it with pointwise operations and compact-open topology i.e. topology of uniform convergence on compact sets.

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Example

$$\widehat{\mathbb{R}^n} \cong \mathbb{R}^n, \quad \widehat{\mathbb{T}^n} \cong \mathbb{Z}^n, \quad \widehat{\mathbb{Z}^n} \cong \mathbb{T}^n, \quad \widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p.$$

The Plancherel Theorem

Fourier transform

Let G be a locally compact abelian group. A Fourier transform of $f \in L^1(G)$ is a map $\hat{f} : G^\wedge \to \mathbb{C}$ defined by the formula

$$\hat{f}(\chi) = \int_G \overline{\chi(x)} f(x) \, d\mu(x).$$

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Theorem

Let G be a locally compact abelian group with a Haar measure μ. Then there exists exactly one Haar measure on $G^\wedge$, called Plancherel measure, such that for all $f \in L^1(G) \cap L^2(G)$ we have

$$\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(G^\wedge)}.$$

Example 1

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- $H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi < +\infty \right\}$

  where $\|\xi\|^2 := \xi_1^2 + \ldots + \xi_n^2$.

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- $$H^s(\mathbb{T}^n) = \left\{ f \in L^2(\mathbb{T}^n) : \sum_{\xi \in \mathbb{Z}^n} \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 < +\infty \right\}$$

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Example 1

- (Zuniga-Galindo, Galeano-Penaloza)

$$H^s(\mathbb{Q}_p^n) = \left\{ f \in L^2(\mathbb{Q}_p^n) : \int_{\mathbb{Q}_p^n} \left(1 + \|\xi\|_p^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi < +\infty \right\}$$

where $\|\xi\|_p$ is a p-adic norm.

Sobolev Spaces

Definition

Let G be a locally compact abelian group, $s \geq 0$ and $\gamma : G^\wedge \to [0, +\infty)$. We say that $f \in L^2(G)$ belongs to Sobolev space $H^s_\gamma(G)$ if

$$\int_{G^\wedge} (1 + \gamma^2(\xi))^s |\hat{f}(\xi)|^2 d\mu(\xi) < \infty.$$

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$$\int\limits_{G^\wedge} (1 + \gamma^2(\xi))^s |\hat{f}(\xi)|^2 d\mu(\xi) < \infty.$$

We equip it with a norm

$$\|f\|_{H^s_\gamma(G)} := \sqrt{\int\limits_{G^\wedge} (1 + \gamma^2(\xi))^s |\hat{f}(\xi)|^2 d\mu(\xi)}.$$

Continuous embeddings 1

Theorem (Górka, Reyes)

Let G be a locally compact abelian group.

- *If $s > \sigma$, then*

$$H^s_\gamma(G) \hookrightarrow H^\sigma_\gamma(G).$$

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- *If $(1+\gamma^2(.))^{-1} \in L^s(G^\wedge)$, then*

$$H^s_\gamma(G) \hookrightarrow C(G).$$

Continuous embeddings 2

Theorem (Reyes, Górka)

Let G be a locally compact abelian group.

- *If* $\alpha > s$ *and* $(1+\gamma^2(.))^{-1} \in L^{\alpha}(G^{\wedge})$, *then*

$$H^s_\gamma(G) \hookrightarrow L^{\alpha^*}(G),$$

where $\alpha^* = \frac{2\alpha}{\alpha - s}$.

Compact embeddings

Theorem (Górka, K., Reyes)

Let G be a compact abelian group.

- *If* $(1+\gamma^2(\cdot))^{-1} \in L^\alpha(G^\wedge)$ *and* $s > \alpha$, *then*
$$H^s_\gamma(G) \hookrightarrow\hookrightarrow C(G).$$
- *If* $(1+\gamma^2(.))^{-1} \in L^\alpha(G^\wedge)$ *and* $s < \alpha$, *then*
$$H^s_\gamma(G) \hookrightarrow\hookrightarrow L^p(G)$$

for all $p < \alpha^* = \frac{2\alpha}{\alpha - s}$.

Example 1 again

- $$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi < +\infty \right\}$$
- $$H^s(\mathbb{T}^n) = \left\{ f \in L^2(\mathbb{T}^n) : \sum_{\xi \in \mathbb{Z}^n} \left(1 + \|\xi\|^2\right)^s \left|\hat{f}(\xi)\right|^2 < +\infty \right\}$$
- $$H^s(\mathbb{Q}_p^n) = \left\{ f \in L^2(\mathbb{Q}_p^n) : \int_{\mathbb{Q}_p^n} \left(1 + \|\xi\|_p^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi < +\infty \right\}$$

Measures of balls

Definition

Let G be a locally compact abelian group and μ a Haar measure on G. We say that a metric $d : G \times G \to \mathbb{R}$ is upper-β regular if:

1. G is not discrete and there exists a constant $D > 0$, such that for all $r > 0$ we have

$$\mu(B(e, r)) \leq D r^{\beta}.$$

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2. G is discrete and there exists $R_0 > 0$ and $D > 0$ such that $B(e, R_0) = \{e\}$ and for $r \geq R_0$

$$\mu(B(e,r)) \leq Dr^{\beta}.$$

The important lemma

Lemma (Górka, K.)

Let G be a locally compact abelian group and $\beta > 0$. Let $\gamma(\xi) = \hat{d}(\xi, \hat{e})$ for all $\xi \in G^\wedge$, where $\hat{d}$ is a metric on $G^\wedge$, which is upper β-regular. Then for all $\alpha > \frac{\beta}{2}$ inequality

$$\left\| \frac{1}{1+\gamma^2(.)} \right\|^{\alpha}_{L^\alpha(G^\wedge)} \leq D(\alpha, \beta),$$

holds, i.e.

$$\left(1+\gamma^2(.)\right)^{-1} \in L^\alpha(G^\wedge).$$

Embeddings into L^p spaces - metric case

Theorem (Górka, K.)

Let G be a locally compact abelian group and $\beta > 0$. Suppose that $\gamma(\xi) = \hat{d}(\xi, \hat{e})$, where $\hat{d}$ is a metric on $G^\wedge$, which is upper β-regular. If $0 < s < \frac{\beta}{2}$, then for all $\alpha > \frac{\beta}{2}$ the following embedding holds

$$H^s_\gamma(G) \hookrightarrow L^{\alpha^*}(G),$$

where $\alpha^* = \frac{2\alpha}{\alpha - s}$.

Dual metrics

Definition

Let G be a locally compact abelian group. We say that metrics $d : G \times G \to \mathbb{R}$ and $\hat{d} : G^\wedge \times G^\wedge \to \mathbb{R}$ are dual metrics if:

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1. d generates the topology of G and $\hat{d}$ generates the compact-open topology of $G^{\wedge}$,
2. for each character $\xi \in G^{\wedge}$ and every $x, y \in G$ we have

$$|\xi(x) - \xi(y)| \leq \hat{d}(\xi, \hat{e}) d(x, y).$$

Embeddings into Hölder spaces

Theorem (Górka, K.)

Let G be a locally compact abelian group such that d and $\hat{d}$ are dual metrics and $\hat{d}$ is upper β-regular. Furthermore, let us assume that $\gamma = \hat{d}$ and that $s = \alpha + \frac{\beta}{2}$ for some $\alpha \in (0,1)$. Then, $H^s_\gamma(G) \hookrightarrow C^{0,\alpha}(G)$. Moreover, there exists $C > 0$ such that inequality

$$\|u\|_{C^{0,\alpha}(G)} \leq C\|u\|_{H^s_\gamma(G)}$$

holds for all $u \in H^s_\gamma(G)$, where

$$\|u\|_{C^{0,\alpha}(G)} = \|u\|_{C(G)} + \sup_{x \neq y \in G} \frac{|u(x) - u(y)|}{d(x,y)^\alpha}.$$

Embeddings into Hölder spaces - p-adic numbers

Theorem (Górka, K.)

Suppose that $s = \alpha + \frac{n}{2}$ *for some* $\alpha \in (0,1)$. *Then*

$$H^s_{\hat{d}}(\mathbb{Q}_p^n) \hookrightarrow C^{0,\alpha}(\mathbb{Q}_p^n).$$

Moreover, there exists $C > 0$ *such that inequality*

$$\|u\|_{C^{0,\alpha}(\mathbb{Q}_p^n)} \leq C\|u\|_{H^s_{\hat{d}}(\mathbb{Q}_p^n)}$$

holds for all $u \in H^s_{\hat{d}}(\mathbb{Q}_p^n)$.

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- For $s > \frac{\beta}{2}$ we have $H^s_\gamma(G) \hookrightarrow C^{0,\alpha}(G)$.

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We already know that

- For $s < \frac{\beta}{2}$ we have $H^s_\gamma(G) \hookrightarrow L^{\alpha*}(G)$.
- For $s > \frac{\beta}{2}$ we have $H^s_\gamma(G) \hookrightarrow C^{0,\alpha}(G)$.
- What happens if $s = \frac{\beta}{2}$?

Trudinger-Moser Inequality

Theorem (Górka, K.)

Let G be a locally compact abelian group. Suppose that $\hat{d}$ is a metric on $G^\wedge$ with a polynomial growth of degree β and that $\gamma = \hat{d}$. Then there exist universal constants $C = C(\beta) > 0, \alpha > 0$ such that

$$\int_G \left(e^{\alpha \left(\frac{u(x)}{\|u\|} \right)^2} - 1 \right) d\mu_G(x) \leq C,$$

where $\|u\| = \|u\|_{H_\gamma^{\frac{\beta}{2}}(G)}$.

Trudinger-Moser Inequality - p-adic numbers

Theorem (Górka, K.)

There exist universal constants $C = C(n) > 0, \alpha > 0$ *such that*

$$\int_{\mathbb{Q}_p^n} \left(e^{\alpha \left(\frac{u(x)}{\|u\|} \right)^2} - 1 \right) d\mu_{\mathbb{Q}_p^n}(x) \leq C,$$

where $\|u\| = \|u\|_{H_{\hat{d}}^{\frac{n}{2}}}$.

Open problems

1. Classify locally compact abelian and metrizable groups for which $H^s_\gamma(G)$, Hajłasz spaces and Newtonian spaces coincide.

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2. What are necessary conditions for which dual metrics exist?
3. Characterize endomorphisms of Sobolev spaces $H^s_\gamma(G)$.

Thank you for your attention

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