Factorization of some Banach function spaces

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Function Spaces XI

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- For any $\varepsilon > 0$ each $z \in L^1$ can be factorized by $x \in E$ and $y \in E'$ in such a way that

$$z = xy \quad \text{and} \quad \|x\|_E \, \|y\|_{E'} \leq (1+\varepsilon)\|z\|_{L^1}.$$

This theorem can be written in the form $L^1 \equiv E \odot E'$, where

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$$E \odot F = \{x \cdot y : x \in E \text{ and } y \in F\}. \tag{1}$$

- Then natural question arises: when is it possible to factorize F through E, that is, when

$$F \equiv E \odot M(E,F)? \ \ (\text{not true in general ! for } F = L^p) \tag{2}$$

Here $M(E,F)$ is the space of multipliers defined as

$$M(E,F) = \{x \in L^0 : yx \in F \text{ for each } y \in E\}$$

with the operator norm $\|x\|_{M(E,F)} = \sup_{\|y\|_E = 1} \|xy\|_F$.

1. Introduction.
2. The space of multipliers $M(E, F)$ and the pointwise product space $E \odot F$.
3. The factorization of Calderón-Lozanovskiĭ spaces.
4. The factorization of symmetric spaces (including the Lorentz and Marcinkiewicz spaces).

Based on the papers:

- Paweł Kolwicz, Karol Leśnik and Lech Maligranda, *Pointwise multipliers of Calderón-Lozanovskiĭ spaces*, Math. Nachr. Vol. 286, no. 8-9, (2013), 876-907.
- Paweł Kolwicz, Karol Leśnik and Lech Maligranda, *Pointwise products of some Banach function spaces and factorization*, J. Funct. Anal. 266, 2, (2014), 616–659.

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  If the Banach ideal space E is considered over the non-atomic measure μ, then we shall say that E is a *Banach function space.*
- The p-convexification $E^{(p)}$ of E is defined by

$$E^{(p)} = \left\{x \in L^0 : |x|^p \in E\right\}, \text{ for } 1 \leq p < \infty, \tag{3}$$

with the norm $\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}$. In case $0 < p < 1$, we will say about p-concavification of E.

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  By a *symmetric function space* (symmetric Banach function space) on I, where $I = [0,1]$ or $I = [0,\infty)$ with the Lebesgue measure m, we mean a Banach ideal space $E = (E, \|\cdot\|_E)$ with the additional property that for any two equimeasurable functions $x \sim y$, $x, y \in L^0(I)$ (that is, $d_x = d_y$, where

$$d_x(\lambda) = m(\{t \in I : |x(t)| > \lambda\}), \lambda \geq 0)$$

  and $x \in E$ we have $y \in E$ and $\|x\|_E = \|y\|_E$. In particular, $\|x\|_E = \|x^*\|_E$, where

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- The fundamental function f_E of a symmetric function space E on I : $f_E(t) = \|\chi_{[0,\, t]}\|_E, t \in I.$

The space of pointwise multipliers.

- Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be ideal Banach spaces in $L^0(\Omega)$. The space of pointwise multipliers $M(E,F)$ is defined by

$$M(E,F) = \{x \in L^0(\Omega) : xy \in F \text{ for all } y \in E\} \tag{4}$$

and the functional on it

$$\|x\|_{M(E,F)} = \sup\{\|xy\|_F,\ y \in E, \|y\|_E \leq 1\} \tag{5}$$

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- $\|\cdot\|_{M(E,F)}$ is a norm and $M(E,F)$ is an ideal Banach space if and only if *supp* $E = \Omega$, that is, E has a *weak unit*, i. e., $x_0 \in E$ such that $x_0 > 0$ μ-a.e. on Ω.

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- If $F = L^1$ we have $M(E, L^1) = E'$, where E' is the classical associated space to E or the Köthe dual space of E.
- Note that $M(E,F)$ can be $\{0\}$.
- It is possible that *supp* $M(E,F)$ is smaller than *supp* $E \cap$ *supp* F.

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- **Theorem**. Let E and F be non-trivial symmetric function spaces on I.
- The space of multipliers $M(E, F)$ is a symmetric function space on I.
- If the symmetric spaces E, F are on $I = [0, 1]$, then $M(E, F) \neq \{0\}$ if and only if $E \hookrightarrow F$.

Pointwise products of some Banach function spaces.

- Given two Banach ideal spaces (real or complex) E and F on (Ω, Σ, μ) define the *pointwise product space* $E \odot F$ as

$$E \odot F = \{x \cdot y : x \in E \text{ and } y \in F\}.$$

with a functional $\|\cdot\|_{E \odot F}$ defined by the formula

$$\|z\|_{E \odot F} = \inf \{\|x\|_E \|y\|_F : z = xy, x \in E, y \in F\}. \qquad (6)$$

The study of spaces $E \odot F$: T. Ando (1960); S. W. Wang (1963); R. O'Neil (1965); P. P. Zabreĭko and Ja. B. Rutickiĭ (1967); G. Dankert (1974); Ja. B. Rutickiĭ (1979); L. Maligranda (1989); M. M. Rao and Z. D. Ren (1991); Y. Raynaud (1992); B. Bollobás and I. Leader (1993); A. Defant, M. Mastyło and C. Michels (2003); S. V. Astashkin and L. Maligranda (2009); T. Kühn and M. Mastyło (2010); A. R. Schep (2010).

- **Proposition.** If E and F are Banach ideal spaces on (Ω, Σ, μ), then $E \odot F$ has an ideal property. Moreover,

$$\begin{aligned} \|z\|_{E\odot F} &= \| \, |z| \, \|_{E\odot F} \\ &= \inf \{ \|x\|_E \|y\|_F : |z| = xy, \, x \in E_+, y \in F_+ \} \\ &= \inf \{ \|x\|_E \|y\|_F : |z| \leq xy, \, x \in E_+, y \in F_+ \} . \end{aligned}$$

- **Definition**. $0 < s < 1$. The Calderón space is defined by

$$E^sF^{1-s} = \{z \in L^0(\Omega) : |z| \leq x^s y^{1-s} \tag{7}$$

for some $x \in E_+,\ y \in F_+\}$ with the norm

$$\|z\|_{E^sF^{1-s}} = \inf\left\{\max\left\{\|x\|_E, \|y\|_F\right\} :\ |z| \leq x^s y^{1-s}, x \in E_+, y \in F_+\right\}. \tag{8}$$

- **Theorem**. Let E and F be a couple of Banach ideal spaces on (Ω, Σ, μ). Then

$$E \odot F \equiv (E^{1/2}F^{1/2})^{(1/2)}, \text{ that is} \tag{9}$$

$$\|z\|_{E \odot F} = \inf\left\{\max\left\{\|x\|_E^2, \|y\|_F^2\right\} : |z| = xy, \|x\|_E = \|y\|_F, x \in E_+, y \in F_+\right\}.$$

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1. Then $E \odot F$ is a quasi-Banach ideal space and the triangle inequality is satisfied with constant 2, i.e.,

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2. If both E and F satisfy the Fatou property, then $E \odot F$ has the Fatou property.

- **Examples**

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  1. If $1 \leq p, q \leq \infty, 1/p + 1/q = 1/r$, then $L^p \odot L^q \equiv L^r$. In particular, $L^p \odot L^p \equiv L^{p/2}$.

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  2. More general, if $1 \leq p, q < \infty, 1/p + 1/q = 1/r$ and E is a Banach ideal space, then $E^{(p)} \odot E^{(q)} \equiv E^{(r)}$.

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2. More general, if $1 \leq p, q < \infty, 1/p + 1/q = 1/r$ and E is a Banach ideal space, then $E^{(p)} \odot E^{(q)} \equiv E^{(r)}$.
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  3. We have $c_0 \odot l^1 \equiv l^\infty \odot l^1 \equiv l^1$ and $c_0 \equiv c_0 \odot l^\infty \neq l^\infty \odot l^\infty \equiv l^\infty$.
- **Theorem**. Suppose that E, F are Banach ideal spaces such that E is p_0-convex with constant 1, F is p_1-convex with constant 1 and $\frac{1}{p_0} + \frac{1}{p_1} \leq 1$. Then $E \odot F$ is a Banach space which is even $\frac{p}{2}$-convex, where

$$\frac{1}{p} = \frac{1}{2}(\frac{1}{p_0} + \frac{1}{p_1}).$$

- **Theorem**. Let E and F be symmetric Banach spaces on $I = (0,1)$ or $I = (0,\infty)$ with the fundamental functions f_E and f_F, respectively. Then $E \odot F$ is a symmetric quasi-Banach space on I and its fundamental function $f_{E\odot F}$ is given by the formula

$$f_{E\odot F}(t) = f_E(t) f_F(t) \quad \text{for } t \in I. \tag{10}$$

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$$f_{E\odot F}(t) = f_E(t) f_F(t) \quad \text{for } t \in I. \tag{10}$$

- In particular $f_E(t) f_{E'}(t) = t = f_{L^1}(t)$ for $t \in I$.

The factorization - several results.

- Let $E = L^{p,1}$ with the norm $\|x\|_E = \frac{1}{p}\int_I t^{\frac{1}{p}-1}x^*(t)\,dt$ for $1 < p < \infty$, then $M(L^{p,1}, L^p) \equiv L^\infty$ and

$$L^{p,1} \odot M(L^{p,1}, L^p) \equiv L^{p,1} \odot L^\infty \equiv L^{p,1} \subsetneq L^p.$$

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  Therefore, we even don't have factorization $L^p = E \odot M(E, L^p)$ with equivalent norms.

- Similarly, if $F = L^{p,\infty}$ with the norm $\|x\|_F = \sup_{t\in I} t^{1/p}x^*(t)$ for $1 < p < \infty$, then $M(L^p, L^{p,\infty}) \equiv M(L^{p',1}, L^{p'}) \equiv L^\infty$ and

  $$L^p \odot M(L^p, L^{p,\infty}) \equiv L^p \odot L^\infty \equiv L^p \subsetneq L^{p,\infty}.$$

  Therefore, again we even don't have equality $F = L^p \odot M(L^p, F)$ with equivalent norms.

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- (P. Nilsson 1985) If F is a Banach ideal space with the Fatou property which is q-concave with constant 1 for $1 < q < \infty$, then

$$F = F'' \equiv L^q \odot M(F', L^{q'}) \equiv L^q \odot M(L^q, F). \qquad (11)$$

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- (A. R. Schep 2010)
  (a) the equivalence in Nilsson theorem.
  (b) A Banach ideal space E with the Fatou property is p-convex with constant $1\,(1 < p < \infty)$ if and only if

$$L^p \equiv E \odot M(E, L^p). \tag{12}$$

Young function

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- Orlicz function
  If additionally $\varphi < \infty$ then φ is called an Orlicz function.

- Define on L^0 a convex modular I_φ by

$$I_\varphi(x) = \begin{cases} \left\| \varphi \circ |x| \right\|_E & \text{if } \varphi \circ |x| \in E, \\ \infty & \text{otherwise}, \end{cases} \tag{13}$$

where $(\varphi \circ |x|)(t) = \varphi(|x(t)|)$, $t \in T$.

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where $(\varphi \circ |x|)(t) = \varphi(|x(t)|)$, $t \in T$.

- By the *Calderón-Lozanovskiĭ space* E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0\} \tag{14}$$

equipped with so called *Luxemburg-Nakano norm* defined by

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\right\}. \tag{15}$$

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- If $E = \Lambda_\omega$ ($e = \lambda_\omega$), then E_φ is the Orlicz-Lorentz function (sequence) space denoted by $(\Lambda_\omega)_\varphi$ ($(\lambda_\omega)_\varphi$) and equipped with the Luxemburg norm.

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- The study of spaces E_φ: A. P. Calderón, G. Ya. Lozanovskiĭ, Z. Altshuler, J. Cerda, P. Foralewski, H. Hudzik, A. Kamińska, L. Maligranda, M. Mastyło, P.K. Lin, V.I. Ovchinnikov,Y. Raynaud, S. Reisner and others.

- For the Young function φ we define its right-continuous inverse in a generalized sense by the formula:

$$\begin{aligned} \varphi^{-1}(v) &= \inf\{u \geq 0 : \varphi(u) > v\} \textit{ for } v \in [0, \infty) \quad (16) \\ \text{and } \varphi^{-1}(\infty) &= \lim_{v \to \infty} \varphi^{-1}(v). \quad (17) \end{aligned}$$

The factorization of Calderón-Lozanovskiĭ spaces.

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- The symbol

$$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$$

for all arguments [for large arguments] (for small arguments) means that there are constants $C, D > 0$ [there are constants $C, D, u_0 > 0$] (there are constants $C, D, u_0 > 0$) such that the inequalities

$$C\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq D\varphi_1^{-1}(u)\varphi_2^{-1}(u) \quad (18)$$

hold for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

Theorem

Let E be a Banach ideal space with the Fatou property and supp $E = \Omega$. Suppose that for two Young functions φ, φ_1 there is a Young function φ_2 such that one of the following conditions holds:

(*i*) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments,
(*ii*) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and $L^\infty \hookrightarrow E$,
(*iii*) $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for small arguments and $E \hookrightarrow L^\infty$.

Then the factorization $E_{\varphi_1} \odot M\left(E_{\varphi_1}, E_\varphi\right) = E_\varphi$ with equivalent norms is valid.

- Applying results concerning the space of multipliers we get

Corollary

Let φ, φ_1 be two Orlicz functions, and let E Banach an ideal space with the Fatou property and supp $E = \Omega$. If the function $f_v(u) = \frac{\varphi(uv)}{\varphi_1(u)}$ is non-increasing on $(0, \infty)$ for any $v > 0$, then the factorization

$$E_{\varphi_1} \odot M(E_{\varphi_1}, E_\varphi) = E_\varphi$$

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- Applying results concerning the space of multipliers we get

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is valid with equivalent norms.

- It is enough to take the function φ_2 as $\varphi_2 = \varphi \ominus \varphi_1 = \sup_{v>0} \{\varphi(uv) - \varphi_1(v)\}$ and used the fact proved in [KLM2013] showing that $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments.

The factorization of symmetric spaces.

- For any quasi-concave function ϕ on I the *Marcinkiewicz function space M_ϕ* is defined by the norm

$$\|x\|_{M_\phi} = \sup_{t \in I} \phi(t)\, x^{**}(t), \quad x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

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- For any concave function ϕ on I the *Lorentz function space Λ_ϕ* given by the norm

$$\|x\|_{\Lambda_\phi} = \int_I x^*(t)d\phi(t) = \phi(0^+)\|x\|_{L^\infty(I)} + \int_I x^*(t)\phi^{'}(t)dt.$$

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- We have

$$\Lambda_{f_E} \overset{1}{\hookrightarrow} E \overset{1}{\hookrightarrow} M_{f_E}. \tag{19}$$

The factorization of symmetric spaces.

- Consider also another Marcinkiewicz space M_ϕ^* than the space M_ϕ as

$$M_\phi^* = M_\phi^*(I) = \{x \in L^0(I) : \|x\|_{M_\phi^*} = \sup_{t \in I} \phi(t)x^*(t) < \infty\}.$$

This Marcinkiewicz space need not be a Banach space and always we have $M_\phi \overset{1}{\hookrightarrow} M_\phi^*$. Moreover, $M_\phi^* \overset{C}{\hookrightarrow} M_\phi$ if and only if

$$\int_0^t \frac{1}{\phi(s)}\, ds \le C \frac{t}{\phi(t)} \text{ for all } \; t \in I. \tag{20}$$

The factorization of symmetric spaces.

- The lower index $p_{\phi,I}$ and upper index $q_{\phi,I}$ of a function ϕ on I are numbers defined as

$$p_{\phi,I} = \lim_{t\to 0^+} \frac{\ln m_{\phi,I}(t)}{\ln t},\ q_{\phi,I} = \lim_{t\to\infty} \frac{\ln m_{\phi,I}(t)}{\ln t},\ m_{\phi,I}(t) = \sup_{s\in I, st\in I} \frac{\phi(st)}{\phi(s)}.$$

It is known that for a quasi-concave function ϕ on $[0,\infty)$ we have $0 \leq p_{\phi,[0,\infty)} \leq p_{\phi,[0,1]} \leq q_{\phi,[0,1]} \leq q_{\phi,[0,\infty)} \leq 1$.

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- We also need for a differentiable increasing function ϕ on I with $\phi(0^+) = 0$ the *Simonenko indices*

$$s_{\phi,I} = \inf_{t\in I} \frac{t\phi'(t)}{\phi(t)},\ \sigma_{\phi,I} = \sup_{t\in I} \frac{t\phi'(t)}{\phi(t)}.$$

  They satisfy $0 \le s_{\phi,I} \le p_{\phi,I} \le q_{\phi,I} \le \sigma_{\phi,I}$.

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  They satisfy $0 \le s_{\phi,I} \le p_{\phi,I} \le q_{\phi,I} \le \sigma_{\phi,I}$.

- The *Boyd indices of E* are defined by

$$\alpha_E = \lim_{s\to 0^+} \frac{\ln \|D_s\|_{E\to E}}{\ln s}, \beta_E = \lim_{s\to\infty} \frac{\ln \|D_s\|_{E\to E}}{\ln s}.$$

The factorization of symmetric spaces.

Theorem

Let ϕ, ψ be a non-decreasing, concave functions on I with $\phi(0^+) = \psi(0^+) = 0$. Suppose $\frac{\psi(t)}{\phi(t)}$ is a non-decreasing function on I.

- If $s_{\phi,I} > 0$ and $s_{\psi,I} > 0$, then

$$\Lambda_\phi \odot M(\Lambda_\phi, \Lambda_\psi) = \Lambda_\psi.$$

Moreover, for any symmetric space F on I with the fundamental function $f_F(t) = \psi(t)$ and under the above assumptions on ϕ and ψ we have

$$\Lambda_\phi \odot M(\Lambda_\phi, F) = F \text{ if and only if } F = \Lambda_\psi. \tag{21}$$

The factorization of symmetric spaces.

Theorem

Let ϕ, ψ be a non-decreasing, concave functions on I with $\phi(0^+) = \psi(0^+) = 0$. Suppose $\frac{\psi(t)}{\phi(t)}$ is a non-decreasing function on I.

- If $\sigma_{\phi,I} < 1$ and $\sigma_{\psi,I} < 1$, then

$$M_\phi \odot M(M_\phi, M_\psi) = M_\psi.$$

Moreover, for any symmetric space E on I having Fatou property, with the fundamental function $f_E(t) = \phi(t)$ and under the above assumptions on ϕ and ψ we have

$$E \odot M(E, M_\psi) = M_\psi \text{ if and only if } E = M_\phi. \tag{22}$$

The factorization of symmetric spaces.

Theorem

Let ϕ, ψ be a non-decreasing, concave functions on I with $\phi(0^+) = \psi(0^+) = 0$. Suppose $\frac{\psi(t)}{\phi(t)}$ is a non-decreasing function on I.

- If $\sigma_{\phi,I} < 1$ and $\sigma_{\psi,I} < 1$, then

$$M_\phi \odot M(M_\phi, M_\psi) = M_\psi.$$

  Moreover, for any symmetric space E on I having Fatou property, with the fundamental function $f_E(t) = \phi(t)$ and under the above assumptions on ϕ and ψ we have

$$E \odot M(E, M_\psi) = M_\psi \text{ if and only if } E = M_\phi. \tag{22}$$

- If $\sigma_{\phi,I} < 1$, $s_{\psi,I} > 0$ and $s_{\psi/\phi,I} > 0$, then

$$M_\phi \odot M(M_\phi, \Lambda_\psi) = \Lambda_\psi. \tag{23}$$

The factorization of symmetric spaces.

Theorem

Let ϕ be an increasing, concave function on I with $0 < p_{\phi,I} \leq q_{\phi,I} < 1$.

- Suppose that F is a symmetric space on I with the lower Boyd index $\alpha_F > q_{\phi,I}$ and such that $M(M_\phi^*, F) \neq \{0\}$. Then

$$F = M_\phi^* \odot M(M_\phi^*, F) = M_\phi \odot M(M_\phi, F).$$

Thank You very much for the attention

The factorization of symmetric spaces.

Theorem

Let ϕ be an increasing, concave function on I with $0 < p_{\phi,I} \leq q_{\phi,I} < 1$.

- Suppose that E is a symmetric space on I with the Fatou property, which Boyd indices satisfy $0 < \alpha_E \leq \beta_E < p_{\phi,I}$ and such that $M(E, \Lambda_\phi) \neq \{0\}$. Then

$$\Lambda_{\phi,1} = E \odot M(E, \Lambda_\phi).$$

- the Lorentz space $\Lambda_{\phi,1}$ on I defined as

$$\Lambda_{\phi,1} = \{x \in L^0(I) : \|x\|_{\Lambda_{\phi,1}} = \int_I x^*(t) \frac{\phi(t)}{t}\, dt < \infty\}.$$

Space $\Lambda_{\phi,1}$ is a Banach space and if $\phi(t) \leq at\phi'(t)$ for all $t \in I$, then $\Lambda_{\phi,1} \overset{1}{\hookrightarrow} \Lambda_\phi \overset{a}{\hookrightarrow} \Lambda_{\phi,1}$.