Smoothness Morrey Spaces of regular distributions, and their envelopes

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Function Spaces XI

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joint work with S. Moura (Coimbra), L. Skrzypczak (Poznań), D. Yang (Beijing) and W. Yuan (Beijing)

Starting point: Morrey Spaces

$0 < p \leq u < \infty$, Morrey space $\mathcal{M}_{u,p}$: $f \in L_p^{\mathrm{loc}}$ with

$$\|f|\mathcal{M}_{u,p}\| = \sup_{x\in\mathbb{R}^n,\ R>0} R^{\frac{n}{u}-\frac{n}{p}} \left(\int_{B(x,R)} |f(y)|^p \,\mathrm{d}y \right)^{1/p} < \infty$$

All spaces are defined on $\mathbb{R}^n$.

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Rem. Ch. Morrey (1938), Campanato, Brudnyi, Peetre (1960's), . . . ,

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Rem. Ch. Morrey (1938), Campanato, Brudnyi, Peetre (1960's), ...,

Some properties:

- $\mathcal{M}_{u,p} = \begin{cases} \{0\}, & p > u \\ L_p, & p = u \end{cases}$ $\curvearrowright$ $u > p$ refined (local) integrability
- $L_u = \mathcal{M}_{u,u} \hookrightarrow \mathcal{M}_{u,p_1} \hookrightarrow \mathcal{M}_{u,p_2}$, $0 < p_2 \le p_1 \le u < \infty$
- $\|f(\lambda\cdot)|\mathcal{M}_{u,p}\| = \lambda^{-n/u}\|f|\mathcal{M}_{u,p}\|$, $\lambda > 0$

All spaces are defined on $\mathbb{R}^n$.

Starting point: Morrey Spaces

A first example

Let $0 < p < u < \infty$.

- $$f(x) = |x|^{-\frac{n}{u}} \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

  $\curvearrowright$ $f \in \mathcal{M}_{u,p}$, but $f \notin L_u$ $\dashrightarrow$ $L_u \subsetneq \mathcal{M}_{u,p}$

- Let $m_k = (2^k, 0, \cdots, 0)$, $k \in \mathbb{N}_0$

  $$\curvearrowright \quad g(x) = \sum_{k=0}^{\infty} f(x - m_k) \ \in \ \mathcal{M}_{u,p} \setminus L_u$$

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Rem. $L_{u,\infty} \subsetneq \mathcal{M}_{u,p}$, $\quad 0 < p < u < \infty$

Classical Smoothness Spaces

Spaces of Besov and Sobolev type

$0 < p \leq \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R},\ \{\varphi_j\}_j$ dyadic partition of unity

$$\left\| f | B^s_{p,q} \right\| = \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p \right\| \right)_j | \ell_q \right\|$$

$$\left\| f | F^s_{p,q} \right\| = \left\| \left\| \left(2^{js} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right)_j | \ell_q \right\| \, | L_p \right\|$$

Rem.

- (classical) Besov spaces $0 < p, q \leq \infty,\ s > n\left(\frac{1}{p} - 1\right)_+$

$$\left\| f | B^s_{p,q} \right\| \sim \| f \mid L_p \| + \left(\int_0^1 \left(\frac{\omega_m(f,t)_p}{t^s} \right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}, \quad m > s$$

- $B^s_{\infty,\infty} = \mathcal{C}^s,\quad s > 0$ Hölder-Zygmund spaces
- $F^s_{p,2} = H^s_p,\ 1 < p < \infty,\ s \in \mathbb{R}$
  $F^k_{p,2} = W^k_p,\ k \in \mathbb{N}_0,\ 1 < p < \infty$ Sobolev spaces

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Approach 1: Smoothness Morrey spaces

The spaces $\mathcal{N}^s_{u,p,q}$ and $\mathcal{E}^s_{u,p,q}$

$0 < p \leq u < \infty$, $q \in (0, \infty]$, $s \in \mathbb{R}$, $\{\varphi_j\}_j$ smooth dyadic res. of unity

(i) Besov-Morrey space $\mathcal{N}^s_{u,p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid \mathcal{N}^s_{u,p,q} \right\| = \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | \mathcal{M}_{u,p} \right\| \right)_j | \ell_q \right\| < \infty$$

(ii) Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}$: $f \in \mathcal{S}'$ with

$$\left\| f \mid \mathcal{E}^s_{u,p,q} \right\| = \left\| \left\| \left(2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_j | \ell_q \right\| \mid \mathcal{M}_{u,p} \right\| < \infty$$

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Rem.

- Kozono/Yamazaki ('94), Mazzucato ('03), Tang/Xu ('05), Sawano ('07-)

H. Kozono and M. Yamazaki
Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data.
Comm. Partial Differential Equations, 19 (1994), 959–1014.

A.L. Mazzucato
Besov-Morrey spaces: function space theory and applications to non-linear PDE.
Trans. Amer. Math. Soc., 355 (2003), 1297–1364 (electronic).

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Rem.

- Kozono/Yamazaki ('94), Mazzucato ('03), Tang/Xu ('05), Sawano ('07-)
- $\mathcal{N}^s_{p,p,q} = B^s_{p,q}$, $\mathcal{E}^s_{p,p,q} = F^s_{p,q}$, $\mathcal{N}^s_{u,p,q} = \mathcal{E}^s_{u,p,q} = \{0\}$ if $p > u$
- elementary properties (quasi-Banach, monotonicity, $\mathcal{S} \hookrightarrow \cdots \hookrightarrow \mathcal{S}'$, ...)
- $\mathcal{E}^0_{u,p,2} = \mathcal{M}_{u,p}$, $1 < p \le u < \infty$, i.e., $\mathcal{E}^0_{p,p,2} = L_p$, $1 < p < \infty$

Approach 2: Smoothness Morrey spaces

From bmo to spaces of type $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$

$$f \in L_1^{\mathrm{loc}}; \quad f_Q = \frac{1}{|Q|} \int_Q f(y)\,\mathrm{d}y, \quad Q \subset \mathbb{R}^n \text{ cubes}$$

$$f \in \mathrm{bmo} \iff$$

$$\|f|\mathrm{bmo}\| = \sup_{|Q|\le 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,\mathrm{d}x + \sup_{|Q|>1} \frac{1}{|Q|} \int_Q |f(x)|\,\mathrm{d}x < \infty$$

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$\mathcal{Q}$ ... collection of all dyadic cubes $Q_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$

$Q \in \mathcal{Q}$ $\dashrightarrow$ side-length 2^{-j_Q}

$\{\varphi_j\}_j$ dyadic res. of unity; Frazier/Jawerth (1990):

$$\|f|\mathrm{bmo}\| \sim \sup_{|Q|\leq 1} \frac{1}{|Q|^{1/2}} \left\| \left\| \left(|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \right)_{j \geq j_Q} |\ell_2 \right\| \mid L_2(Q) \right\|$$

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The spaces $B^{s,\tau}_{p,q}$ and $F^{s,\tau}_{p,q}$

$0 < p, q \le \infty$, $\tau \in [0,\infty)$, $s \in \mathbb{R}$, $\{\varphi_j\}_j$ dyadic res. of unity

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Rem.

- El Baraka ('02), Yuan/Sickel/Yang ('10-)

W. Yuan, W. Sickel and D. Yang,
Morrey and Campanato Meet Besov, Lizorkin and Triebel.
Lecture Notes in Mathematics 2005, Springer, Berlin, 2010.

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- $\mathrm{bmo} = B^{0,1/2}_{2,2} = F^{0,1/p}_{p,2}$, $0 < p < \infty$

Approach 2': Smoothness Morrey spaces

Triebel's Hybrid Spaces

$0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-\frac{n}{p} \leq r < \infty$

global Besov-Morrey and Triebel-Lizorkin-Morrey spaces

$$\|f|L^r B^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |Q_{J,M}|^{-(\frac{1}{p}+\frac{r}{n})} \inf_{g\in\mathrm{span}\{\Psi^J_m, m\in\mathbb{Z}^n\}} \|f-g|B^s_{p,q}(2Q_{J,M})\|$$

$$\|f|L^r F^s_{p,q}\| = \sup_{J\in\mathbb{Z}, M\in\mathbb{Z}^n} |Q_{J,M}|^{-(\frac{1}{p}+\frac{r}{n})} \inf_{g\in\mathrm{span}\{\Psi^J_m, m\in\mathbb{Z}^n\}} \|f-g|F^s_{p,q}(2Q_{J,M})\|$$

Rem.

- Triebel: local spaces ('13), hybrid spaces ('14)

H. Triebel
Local function spaces, heat and Navier-Stokes equations.
EMS Tracts in Mathematics 20, EMS Publishing House, Zürich, 2013.

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Hybrid Function Spaces, Heat and Navier-Stokes Equations,
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- $L^r B^s_{p,q} = B^{s,\tau}_{p,q}$, $L^r F^s_{p,q} = F^{s,\tau}_{p,q}$, $\tau = \frac{1}{p} + \frac{r}{n}$

Approach 2': Smoothness Morrey spaces

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Comparison: Smoothness Morrey spaces

Relation between the different scales

- $\tau \geq \frac{1}{p}$ (with $q=\infty$ if $\tau=\frac{1}{p}$): $B^{s,\tau}_{p,q} = F^{s,\tau}_{p,q} = B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}$
- $F^{s,\tau}_{p,q} = \mathcal{E}^{s}_{u,p,q}$ with $\tau=\frac{1}{p}-\frac{1}{u}$, $0<p\leq u<\infty$
- $\mathcal{N}^{s}_{u,p,q} \hookrightarrow B^{s,\tau}_{p,q}$ with $\tau=\frac{1}{p}-\frac{1}{u}$

  coincidence (only) if $\tau=0$ or $q=\infty$, i.e., $\mathcal{N}^{s}_{u,p,\infty} = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}$

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W. Sickel,
Smoothness spaces related to Morrey spaces – a survey.
Part I: Eurasian Math. J. 3 (2012), 110-149;
Part II: Eurasian Math. J. 4 (2013), 82-124.

Introduction

Smoothness Spaces of Morrey Type
- Different approaches
- Comparison between the different approaches

Continuity envelopes
- Concept and basic examples
- Results for Smoothness Morrey spaces
- Application: Approximation numbers

Growth envelopes
- Concept and basic examples
- Results for Smoothness Morrey spaces

Continuity envelopes

The concept

$$\omega(f,t) = \sup_{|h|\leq t} \sup_{x\in\mathbb{R}^n} |f(x+h) - f(x)| \ , \quad t>0$$

Definition 1

$X \hookrightarrow \mathrm{C}$, continuity envelope function $\mathcal{E}_{\mathsf{C}}^{X}:(0,\infty) \to [0,\infty)$

$$\mathcal{E}_{\mathsf{C}}^{X}(t) \sim \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t} \ , \quad t>0$$

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Proposition 2

(i) $X \hookrightarrow \mathrm{Lip}^1 \iff \sup_{t>0} \mathcal{E}_{\mathsf{C}}^{X}(t) < \infty$

(ii) $X_1 \hookrightarrow X_2 \implies \mathcal{E}_{\mathsf{C}}^{X_1}(t) \leq c\,\mathcal{E}_{\mathsf{C}}^{X_2}(t), \quad t>0$

Rem. 'fine index' u_{C}^{X} $\curvearrowright$ $\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^{X}, u_{\mathsf{C}}^{X}\right)$ **continuity envelope**

Continuity envelopes

Example: $B^s_{p,q}$

Proposition 3

If $t \to 0$, then

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t) \sim \begin{cases} |\log t|^{\frac{1}{q'}}, & s = \frac{n}{p}+1, \quad q > 1 \\ t^{-(\frac{n}{p}+1-s)}, & \frac{n}{p} < s < \frac{n}{p}+1 \\ t^{-1}, & s = \frac{n}{p}, \quad 0 < q \le 1 \end{cases}$$

Continuity envelopes of Smoothness Morrey spaces

Preparation

$\mathcal{E}_C^X$ *reasonable* if $X \hookrightarrow C$, *interesting* if $X \not\hookrightarrow \mathrm{Lip}^1$

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Proposition 4

(i) $s \in \mathbb{R}$, $0 < p < u < \infty$, $q \in (0, \infty]$

$$\mathcal{N}^s_{u,p,q} \hookrightarrow C \iff \begin{cases} s > \frac{n}{u}, & \text{or} \\ s = \frac{n}{u} & \text{and} \quad q \in (0,1] \end{cases}$$

$$\mathcal{N}^s_{u,p,q} \not\hookrightarrow \mathrm{Lip}^1 \iff \begin{cases} s < \frac{n}{u} + 1, & \text{or} \\ s = \frac{n}{u} + 1 & \text{and} \quad q \in (1,\infty], \end{cases}$$

(ii) $s \in \mathbb{R}$, $p, q \in (0, \infty]$, $\tau > 0$

$$B^{s,\tau}_{p,q} \hookrightarrow C \iff s > n\left(\frac{1}{p} - \tau\right)$$

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Continuity envelopes of Smoothness Morrey spaces

Preparation

$\mathcal{E}_C^X$ *reasonable* if $X \hookrightarrow C$, *interesting* if $X \not\hookrightarrow \mathrm{Lip}^1$

Proposition 4

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Continuity envelopes of Smoothness Morrey spaces

The spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^n)$

Theorem 5

(i) Let $0 < p < u < \infty$. If $\mathcal{N}^s_{u,p,q} \hookrightarrow C$ and $\mathcal{N}^s_{u,p,q} \not\hookrightarrow \mathrm{Lip}^1$, then

$$\mathcal{E}_{\mathsf{C}}^{\mathcal{N}^s_{u,p,q}}(t) \sim \mathcal{E}_{\mathsf{C}}^{B^s_{u,q}}(t), \quad t \to 0$$

Rem.

- $\frac{n}{u} \le s \le \frac{n}{u} + 1$ (with additional assumptions in limiting cases)
- parallel results for spaces $\mathcal{E}^s_{u,p,q}$
- Yuan/H./Moura/Skrzypczak/Yang ('15)

Continuity envelopes of Smoothness Morrey spaces

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(ii) Let $\tau > 0$. If $t \to 0$, then

$$\mathcal{E}_{\mathsf{C}}^{B^{s,\tau}_{p,q}}(t) \sim \begin{cases} t^{s+n(\tau-\frac{1}{p})-1}, & 0 < s - n\left(\frac{1}{p}-\tau\right) < 1 \\ |\log t|, & s - n\left(\frac{1}{p}-\tau\right) = 1 \end{cases}$$

Rem.

- $\frac{n}{u} \leq s \leq \frac{n}{u} + 1$ (with additional assumptions in limiting cases)
- parallel results for spaces $\mathcal{E}^s_{u,p,q}$ and $F^{s,\tau}_{p,q}$
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Continuity envelopes in Smoothness Morrey spaces

Main tools in the proofs

- embeddings into 'classical' spaces, like

$$B^{\sigma}_{r,\infty} \hookrightarrow B^{s,\tau}_{p,q} \hookrightarrow B^{s+n(\tau-\frac{1}{p})}_{\infty,\infty} \quad \text{with} \quad \sigma = s + n\left(\tau - \frac{1}{p}\right) + \frac{n}{r}$$

- embeddings between different scales, in particular

$$F^{s,\tau}_{p,q} = \mathcal{E}^{s}_{u,p,q}, \quad \mathcal{N}^{s}_{u,p,\infty} = B^{s,\tau}_{p,\infty} \quad \text{if} \quad \tau = \frac{1}{p} - \frac{1}{u},\ 0 < p \le u < \infty$$

- limiting embeddings of Sobolev and Franke-Jawerth type, like

$$\mathcal{N}^{s_1}_{u_1,p_1,q_1} \hookrightarrow \mathcal{E}^{s}_{u,p,q} \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2} \quad \text{with} \quad s_i - \frac{n}{u_i} = s - \frac{n}{u}$$

[H./Skrzypczak ('14)]

$$B^{s_1,\tau_1}_{p_1,q_1} \hookrightarrow F^{s,\tau}_{p,q} \hookrightarrow B^{s_2,\tau_2}_{p_2,q_2} \quad \text{with} \quad s_i - \frac{n}{p_i} + n\tau_i = s - \frac{n}{p} + n\tau$$

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Application

Approximation numbers

$\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $\quad B^{s,\tau}_{p,q}(\Omega)$, $\mathcal{N}^s_{u,p,q}(\Omega)$ defined by restriction

approximation numbers of $T \in \mathcal{L}(X)$

$$a_k(T) = \inf\{\|T - S\| : S \in \mathcal{L}(X), \operatorname{rank} S < k\}, \quad k \in \mathbb{N}$$

Rem. connections to spectral theory, eigenvalue estimates

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Rem. connections to spectral theory, eigenvalue estimates

Example $p, q \in (0, \infty]$, $s > \frac{n}{p}$

$$a_k\left(\mathrm{id}_\Omega : B^s_{p,q}(\Omega) \hookrightarrow C(\Omega)\right) \sim \begin{cases} k^{-\frac{s}{n}+\frac{1}{p}}, & \text{if } p \in [2,\infty], \\ k^{-\frac{s}{n}+\frac{1}{2}}, & \text{if } p \in (1,2),\ s > n, \\ k^{-\left(\frac{s}{n}-\frac{1}{p}\right)\frac{p'}{2}}, & \text{if } p \in (1,2),\ s < n \end{cases}$$

Application

Approximation numbers

Corollary 6

- $u \in [2, \infty)$, $p \in (0, u]$, $q \in (0, \infty]$, $\frac{n}{u} < s < \frac{n}{u} + 1$

$$a_k \left(\mathrm{id} : \mathcal{N}^s_{u,p,q}(\Omega) \to C(\Omega)\right) \sim k^{-\frac{s}{n}+\frac{1}{u}}, \quad k \in \mathbb{N}$$

- $p \in [2, \infty]$, $q \in (0, \infty]$, $\tau \geq 0$, $n(\frac{1}{p} - \tau) < s < n(\frac{1}{p} - \tau) + 1$

$$a_k \left(\mathrm{id} : B^{s,\tau}_{p,q}(\Omega) \to C(\Omega)\right) \sim k^{-\frac{s}{n}-\tau+\frac{1}{p}}, \quad k \in \mathbb{N}$$

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Rem.

- $a_k\, (\mathrm{id} : X(\Omega) \longrightarrow C(\Omega)) \;\leq\; c\; k^{-\frac{1}{n}}\; \mathcal{E}^X_{\mathsf{C}} \left(k^{-\frac{1}{n}}\right), \quad k \in \mathbb{N}$
- parallel results for $\mathcal{E}^s_{u,p,q}(\Omega)$, $F^{s,\tau}_{p,q}(\Omega)$
- H./Skrzypczak ('12): more general criteria for the compactness of related embeddings

Growth envelopes

The concept

$$f^*(t) = \inf\left\{s \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leq t\right\}, \quad t \geq 0$$

Definition 7

$X \subset L_1^{\mathrm{loc}}$, ***growth envelope function*** $\mathcal{E}_{\mathsf{G}}^X : (0,\infty) \to [0,\infty]$

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup_{\|f|X\| \leq 1} f^*(t), \quad t > 0$$

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Proposition 8

(i) $X \hookrightarrow L_\infty \iff \sup_{t>0} \mathcal{E}_\mathsf{G}^X(t) < \infty$

(ii) $X_1 \hookrightarrow X_2 \implies \mathcal{E}_\mathsf{G}^{X_1}(t) \leq c\, \mathcal{E}_\mathsf{G}^{X_2}(t), \quad t > 0$

(iii) X rearrangement-invariant with fundamental function φ_X

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Rem. additional 'fine index' u_{G}^X $\curvearrowright$ $\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X, u_{\mathsf{G}}^X\right)$ 'growth envelope'

Example $\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = \left(t^{-\frac{1}{p}}, q\right), \quad 0 < p < \infty,\ 0 < q \leq \infty$

Growth envelopes in Morrey spaces

The bottom line

Proposition 9

$0 < p < u < \infty$

$$\mathcal{E}_{\mathsf{G}}^{\mathcal{M}_{u,p}(\mathbb{R}^n)}(t) = \infty, \quad t > 0$$

Rem.

- $0 < p = u < \infty$: $\mathcal{E}_{\mathsf{G}}^{L_p(\mathbb{R}^n)}(t) \sim t^{-1/p}$, $t > 0$
- H./Moura ('14), idea: consider

$$f(x) = |x|^{-\frac{n}{u}} \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$g(x) = \sum_{k=0}^{\infty} f(x - m_k), \quad m_k = (2^k, 0, \cdots, 0)$$

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Growth envelopes in Smoothness Morrey spaces

Preparation

$\mathcal{E}_{\mathsf{G}}^{X}$ *reasonable* for $X \subset L_1^{\mathrm{loc}}$, *interesting* for $X \not\hookrightarrow L_\infty$

Growth envelopes in Smoothness Morrey spaces

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$\curvearrowright$ When do $\mathcal{N}^s_{u,p,q}$, $B^{s,\tau}_{p,q}$ contain *regular* distributions only?

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$s \in \mathbb{R}$, $\tau > 0$, $0 < q \leq \infty$, $0 < p < u < \infty$

Rem.

- additional conditions in limiting cases, parallel results for $\mathcal{E}^s_{u,p,q}$, $F^{s,\tau}_{p,q}$
- classical ($u = p$, $\tau = 0$): Sickel/Triebel (1995)
- Morrey: H./Moura/Skrzypczak ('15)

Growth envelopes in Smoothness Morrey spaces

Preparation, II

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$s \in \mathbb{R}$, $\tau > 0$, $0 < q \le \infty$, $0 < p < u < \infty$

Growth envelopes in Smoothness Morrey spaces

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Rem. H./Skrzypczak ('13, '14), Yuan/H./Skrzypczak/Yang ('15), Yuan/H./Moura/Skrzypczak/Yang ('15)

Growth envelopes in Smoothness Morrey spaces

Example: $B^s_{p,q}$

Example $0<p<\infty,\ 0<q\leq\infty,\ \ s>\sigma_p=n\left(\frac{1}{p}-1\right)_+$

(i) For $s<\frac{n}{p}$, $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t)\ \sim\ t^{-\frac{1}{p}+\frac{s}{n}},\quad t\to 0$

(ii) If $s=\frac{n}{p}$, $1<q\leq\infty$, then

$$\mathcal{E}_{\mathsf{G}}^{B^{n/p}_{p,q}}(t)\ \sim\ |\log t|^{\frac{1}{q'}},\quad t\to 0$$

(iii) We have $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t)\ \sim\ t^{-\frac{1}{p}},\quad t\to\infty$

Growth envelopes in Smoothness Morrey spaces

Example: $B^s_{p,q}$

Example $0 < p < \infty$, $0 < q \leq \infty$, $s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+$

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Example: $B^s_{p,q}$

Example $0 < p < \infty,\ 0 < q \leq \infty,\ \ s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+$

(i) For $s < \frac{n}{p}$, $\mathcal{E}_{\mathrm{G}}^{B^s_{p,q}}(t) \sim t^{-\frac{1}{p} + \frac{s}{n}}, \quad t \to 0$

(ii) If $s = \frac{n}{p}$, $1 < q \leq \infty$, then

$$\mathcal{E}_{\mathrm{G}}^{B^{n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{q'}}, \quad t \to 0$$

(iii) We have $\mathcal{E}_{\mathrm{G}}^{B^s_{p,q}}(t) \sim t^{-\frac{1}{p}}, \quad t \to \infty$

Growth envelopes in Smoothness Morrey spaces

The case $\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)$

Theorem 10

Let $0 < p < u < \infty$. If $\mathcal{N}^s_{u,p,q} \subset L_1^{\mathrm{loc}}$ and $\mathcal{N}^s_{u,p,q} \not\hookrightarrow L_\infty$, then

$$\mathcal{E}_{\mathsf{G}}^{\mathcal{N}^s_{u,p,q}(\mathbb{R}^n)}(t) = \infty, \quad t > 0.$$

Rem.

- $0 < p = u < \infty$

$$\mathcal{E}_{\mathsf{G}}^{\mathcal{N}^s_{u,p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim \begin{cases} t^{-\frac{1}{p}+\frac{s}{n}}, & s < \frac{n}{p} \\ |\log t|^{1/q'}, & s = \frac{n}{p},\ q > 1 \end{cases}$$

- $\frac{p}{u}\sigma_p \le s \le \frac{n}{u}$ (with additional conditions in endpoints)
- parallel results for $\mathcal{E}^s_{u,p,q}$; essential again: $\mathbb{R}^n$
- H./Moura ('14), H./Moura/Skrzypczak ('15)
  idea: atomic decomposition (with moment conditions)

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Growth envelopes in Smoothness Morrey spaces

The case $B^{s,\tau}_{p,q}(\mathbb{R}^n)$

Corollary 11

Let $\tau > 0$, $s \leq n\left(\frac{1}{p} - \tau\right)$, with $B^{s,\tau}_{p,q} \subset L_1^{\mathrm{loc}}$.

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Let $\tau > 0$, $s \leq n\left(\frac{1}{p} - \tau\right)$, with $B^{s,\tau}_{p,q} \subset L_1^{\mathrm{loc}}$. If $0 < \tau < \frac{1}{p}$ or $\tau = \frac{1}{p}$, $s = 0$, $0 < p = q \leq 2$, then

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Rem.

- $\tau = 0$: $\quad \mathcal{E}_{\mathrm{G}}^{B^{s,\tau}_{p,q}}(t) \sim \mathcal{E}_{\mathrm{G}}^{B^{s}_{p,q}}(t) < \infty, \quad t > 0$
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  idea: atomic decomposition & corresponding results for $\mathcal{N}^s_{u,p,q}$
- $p = q = 2$, $s = 0$, $\tau = \frac{1}{2}$ $\curvearrowright$ $\mathrm{bmo}(\mathbb{R}^n) = B^{0,1/2}_{2,2}(\mathbb{R}^n)$
  $\curvearrowright$ $\mathcal{E}_{\mathrm{G}}^{\mathrm{bmo}(\mathbb{R}^n)}(t) = \infty, \quad t > 0$ (Triebel '01)

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Thank you very much for your attention!