Non-smooth atomic decomposition of 2-microlocal spaces with variable integrability

Helena Gonçalves[1]

Chemnitz University of Technology

Function Spaces XI

July 6-10, 2015, Zielona Góra

[1] joint work with Henning Kempka

Decomposition of function spaces

Idea

Represent functions as linear combination of basic functions

$$f \text{ in some function space } X \qquad \Longleftrightarrow \qquad f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

where...

$\lambda_{\nu m}$ $\rightsquigarrow$ elements of a certain sequence space

$a_{\nu m}$ $\rightsquigarrow$ building blocks (e.g. atoms, molecules, ...)

Non-smooth atomic decomposition

- **Authors:** Triebel, Moura, Piotrowska, Piotrowski, Caetano, Lopes $\rightsquigarrow$ Besov spaces with $p = q$
- **Schneider and Vybiral (2011), Scharf (2013)** $\rightsquigarrow$ Classical Besov spaces with $p \neq q$

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Class of variable exponents

- $\mathcal{P}(\mathbb{R}^n) = \{p : \mathbb{R}^n \to (0, \infty]$ mesurable, bounded away from $0\}$
- $\mathbb{R}^n_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\}$
- $p^- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$
- Let $f : \mathbb{R}^n \to \mathbb{C}$ measurable.

$$L_{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) < \infty, \text{ for some } \lambda > 0 \right\}$$

where

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} \mathrm{d}x + \operatorname{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|.$$

- $\|f|L_{p(\cdot)}(\mathbb{R})\| := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1 \right\}$

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Almeida, Hästö (2010):

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf \left\{ \lambda_\nu > 0 : \varrho_{p(\cdot)} \left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}} \right) \leq 1 \right\}$$

If $q^+ < \infty$, then we have the simpler expression

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \left\| |f_\nu|^{q(\cdot)} \mid L_{\frac{p(\cdot)}{q(\cdot)}} \right\|.$$

$$\|f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| = \inf \left\{ \mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left(\frac{f_\nu}{\mu} \right) \leq 1 \right\}.$$

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Regularity conditions

Definition

Let $g \in C(\mathbb{R}^n)$.

(i) g is *locally* $\log$-*Hölder continuous*, $g \in C^{\log}_{loc}(\mathbb{R}^n)$ if

$$\exists c_{\log} > 0 : \quad |g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x-y|)}, \quad \forall x, y \in \mathbb{R}^n.$$

(ii) g is *globally* $\log$-*Hölder continuous*, $g \in C^{\log}(\mathbb{R}^n)$ if $g \in C^{\log}_{loc}(\mathbb{R}^n)$ and

$$\exists g_\infty \in \mathbb{R} : \quad |g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

(iii) $\mathcal{P}^{\log}(\mathbb{R}^n) := \{p \in \mathcal{P}(\mathbb{R}^n) : 1/p \in C^{\log}(\mathbb{R}^n)\}$.

Smooth dyadic resolution of unity

- Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$:
  (i) $\varphi_0(x) = 1$ if $|x| \leq 1$
  (ii) $\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$
- We define $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and $\varphi_j(x) := \varphi(2^{-j}x), \forall j \in \mathbb{N}$. Then,

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

The sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ forms a *smooth dyadic resolution of unity*.

Admissible weight sequence

Definition

Let $\alpha \geq 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$. A sequence of non-negative measurable functions in $\mathbb{R}^n$, $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0}$, belongs to $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ if:

(i) $\exists C > 0: \quad 0 < w_j(x) \leq C w_j(y)(1 + 2^j|x-y|)^{\alpha}, \quad j \in \mathbb{N}_0, x, y \in \mathbb{R}^n;$

(ii) $\forall j \in \mathbb{N}_0, \quad 2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x), \quad x \in \mathbb{R}^n.$

Such a system $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ is called *admissible weight sequence*.

Definition

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Then $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_{\varphi} := \|(w_j\,(\varphi_j \widehat{f})^{\vee})_{j\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\| < \infty.$$

Definition

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$, $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$.Then $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

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Smooth atoms

Let $K, L \in \mathbb{N}_0$ and $d > 1$. A K-times continuously differentiable complex-valued function $a \in C^K(\mathbb{R}^n)$ is called a $[K, L]$-*atom centered at* Q_{jm}, $j \in \mathbb{N}_0, m \in \mathbb{Z}^n$, if:

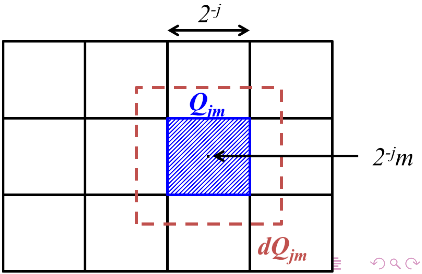
(i) $\operatorname{supp} a \subset dQ_{jm}$ *(compact support)*

(ii) $|D^\beta a(x)| \leq 2^{|\beta| j}, \quad |\beta| \leq K$ *(smoothness K)*

(iii) $\int_{\mathbb{R}^n} x^\beta a(x) \mathrm{d}x = 0, \quad 0 \leq |\beta| < L, j \geq 1.$ *(moment conditions)*

Q_{jm} $\rightsquigarrow$ cube in $\mathbb{R}^n$

$\chi_{jm}(x)$ $\rightsquigarrow$ characteristic function of Q_{jm}



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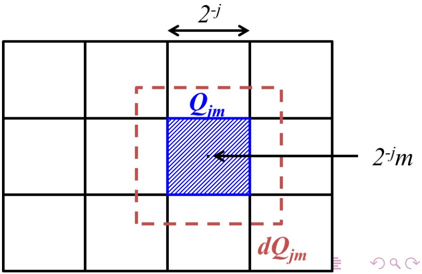

Theorem [Drihem (2013) for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$]

Let $\boldsymbol{w} = (w_\nu)_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Furthermore, let $d > 1$, $K, L \in \mathbb{N}_0$ with

$$K > \alpha_2 \quad \text{and} \quad L > \sigma_p - \alpha_1 + c_{log}(1/q)$$

be fixed. Then every $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if an only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}, \quad \text{convergence being in } \mathcal{S}'(\mathbb{R}^n),$$

for $(a_{\nu m})_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ smooth $[K, L]$-atoms and $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover,

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| \sim \inf \|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|,$$

where the infimum is taken over all possible representations of f.

$$\|\lambda \mid b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \left\| \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\, w_\nu(2^{-\nu}m)\, \chi_{\nu m} \right)_{\nu\in\mathbb{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \right\|$$

Theorem [Kempka (2010)]

Let $\boldsymbol{w} = (w_\nu)_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Furthermore, let $d > 1$, $K, L \in \mathbb{N}_0$ with

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Let $s > 0$ and $s = \lfloor s \rfloor^- + \{s\}^+ \qquad (\lfloor s \rfloor^- \in \mathbb{Z}, \{s\}^+ \in (0,1])$.

Hölder space $\mathcal{C}^s$: $\quad f \in C^{\lfloor s \rfloor^-}$ with

$$\|f \mid \mathcal{C}^s\| := \sum_{|\alpha| \leq \lfloor s \rfloor^-} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| + \sum_{|\alpha| = \lfloor s \rfloor^-} \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\{s\}^+}} < \infty.$$

If $s = 0$, then we set $\mathcal{C}^0 = L_\infty$.

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Let $K, L \geq 0$, $d > 1$ and $c > 0$. A function $a : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called a non-smooth $[K, L]$-atom centered at $Q_{\nu m}$, for all $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if

(i) $\operatorname{supp} a \subset d\, Q_{\nu m}$,

(ii) $\|a(2^{-\nu} \cdot) \mid \mathcal{C}^K\| \leq c$, *(Triebel, Winkelvoß)*

(iii) and for every $\psi \in \mathcal{C}^L$ it holds *(Skrzypczak)*

$$\left| \int_{d\, Q_{\nu m}} \psi(x) a(x) dx \right| \leq c\, 2^{-\nu(L+n)} \|\psi \mid \mathcal{C}^L\|.$$

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Every smooth $[K, L]$-atom is a non-smooth $[K, L]$-atom.

Theorem

The atomic representation theorem for $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is valid with non-smooth $[K, L]$-atoms. Hereby,

$$K > \alpha_2 \quad \textit{and} \quad L > \sigma_p - \alpha_1 + c_{log}(1/q)$$

resp.

$$p^+, q^+ < \infty, \quad K > \alpha_2 \quad \textit{and} \quad L > \sigma_{p,q} - \alpha_1.$$

The problem

We want to observe the behaviour of the mapping

$$f \longmapsto \varphi \cdot f$$

where...

$f \quad \rightsquigarrow \quad$ element of $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ or $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

$\varphi \quad \rightsquigarrow \quad$ suitable smooth function

Idea

$$f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \qquad \longrightarrow \qquad f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}$$

Multiplication by φ ... $\qquad \varphi \cdot f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, (\varphi \cdot a_{\nu m})$

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Multiplication by φ ... $\quad \varphi \cdot f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, (\varphi \cdot a_{\nu m})$

Lemma [Scharf (2013)]

There exists a constant c with the following property: for all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, all non-smooth $[K, L]$-atoms $a_{\nu m}$ and all $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ with $\rho \geq \max(K, L)$, the product

$$c \left\| \varphi \mid \mathcal{C}^\rho(\mathbb{R}^n) \right\|^{-1} \cdot \varphi \cdot a_{\nu m}$$

is a non-smooth $[K, L]$-atom with support in $d\, Q_{\nu m}$.

Theorem

Let $\boldsymbol{w} = (w_j)_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1, \alpha_2}(\mathbb{R}^n)$ and $p, q \in \mathcal{P}^{log}(\mathbb{R}^n)$.

(i) *Let $\rho > \max\left(\alpha_2, \sigma_p - \alpha_1 + c_{log}(1/q)\right)$. There exists a positive number c such that*

$$\left\| \varphi \cdot f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \leq c \left\| \varphi \mid \mathcal{C}^\rho(\mathbb{R}^n) \right\| \cdot \left\| f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|$$

for all $\varphi \in \mathcal{C}^\rho(\mathbb{R}^n)$ and all $f \in B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Let $p^+, q^+ < \infty$ and $\rho > \max(\alpha_2, \sigma_{p,q} - \alpha_1)$. There exists a positive number c such that*

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