Function Spaces XI

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Invariant means in the theory of functional equations and inequalities

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Theorem (S. Banach, S. Mazur). **Let T be a nonempty set and let Φ be a function space of mappings $\varphi : T \longrightarrow T$ such that**

$$x \circ \varphi \in B(T, \mathbb{R}) \text{ for all } x \in B(T, \mathbb{R}).$$

Then there exists a positive linear functional $m : B(T, \mathbb{R}) \longrightarrow \mathbb{R}$ such that $m(\chi_T) = 1$ and

$$m(x \circ \varphi) = m(x) \text{ for } x \in B(T, \mathbb{R}),\ \varphi \in \Phi,$$

if and only if

$$\sup \left\{ \sum_{i=1}^{n} (x_i(t) - x_i \circ \varphi_i(t)) : t \in T \right\} \geq 0$$

for every $x_i \in B(T, \mathbb{R})$, $\varphi_i \in \Phi$, $i = 1, ..., n$, $n \in \mathbb{N}$.

In particular, the latter condition is satisfied provided that the family Φ consists of commuting functions.

THEOREM (J. VON NEUMANN). **Let** $(G,+)$ **be a locally compact group admitting a (bilaterally) invariant mean and let** λ **stands for the left (right) Haar measure on the group** $(G,+)$**. Then there exists a finitely additive extension** $\mu : 2^G \longrightarrow [0,\infty]$ **of the measure** λ **onto the power set of** G **such that**

$$\mu(x+A) = \mu(A) \textbf{ for } x \in G,\ A \subset G,$$

(resp.:

$$\mu(A+x) = \mu(A) \textbf{ for } x \in G,\ A \subset G\ ).$$

In the sequel, by an **amenable** semigroup we shall understand any semigroup admitting an invariant mean.

Now, given an amenable group (with a bilaterally invariant mean m) put

$$\mathcal{N} := \{A \subset G : \, m(\chi_A) = 0\} \ .$$

Then one has:

1. $\emptyset \in \mathcal{N}$ **and** $G \notin \mathcal{N}$
2. $A, B \in \mathcal{N} \Longrightarrow A \cup B \in \mathcal{N}$
3. $(A \in \mathcal{N}$ **and** $B \subset A) \Longrightarrow B \in \mathcal{N}$
4. $(A \in \mathcal{N}$ **and** $x \in G) \Longrightarrow x + A \in \mathcal{N}$ **and** $A + x \in \mathcal{N}$.

If $(G, +)$ is a group and a family $\mathcal{N} \subset 2^G$ enjoyes the properties 1. - 4. along with

5. for every $A \in \mathcal{N}$ one has $-A \in \mathcal{N}$,

then the family $\mathcal{N}$ is used to be called a proper linearly invariant set ideal (in short: a p.l.i. ideal).

Lemma. **Let** $(G, +)$ **a group and let** $\mathcal{R} \subset 2^G$ **be a nonempty collection of sets hereditary with respect to descending inclusions and such that for every** $E \in \mathcal{R}$ **no finite union of the sets of the form**

$$x + (E \cup (-E)) + y, \; x, y \in G,$$

coincides with G**. Then** $\mathcal{R}$ **is contained in a p.l.i ideal.**

Corollary 1. **Let $(G, +)$ be a group endowed with a translation invariant metric such that $\operatorname{diam} G = \infty$. Then any set $E \subset G$ such that $\operatorname{diam} E < \infty$ yields a member of some p.l.i. ideal in $(G, +)$.**

Corollary 2. **In any nontrivial normed linear space $(X, \|\cdot\|)$ each bounded set yields a member of some p.l.i. ideal of subsets of X.**

Corollary 3. **Let $(G, +)$ be an amenable group and let $H \subset G$ be its subgroup with infinite index. Then H yields a member of some p.l.i ideal of subsets of G.**

Corollary 4. **In any real linear space each linear variety (in particular, each proper linear subspace) yields a member of some p.l.i. ideal.**

Corollary 5. **For each nontrivial linear functional f on a real or complex linear space X every set of the form $f^{-1}(K)$, where K stands for a bounded subset of the field in question, yields a member of some p.l.i. ideal of subsets of X.**

Let $(G, +)$ be an Abelian Polish topological group and let $\mathcal{A}_\mu(G)$ denotes the completion of the σ-field $\mathcal{B}(G)$ of all Borel sets with respect to a given measure $\mu : \mathcal{B}(G) \longrightarrow [0, \infty]$. Let further

$$\mathcal{A}(G) := \bigcap \{ \mathcal{A}_\mu(G) : \ \mu \text{ is a probabilistic measure on } \mathcal{B}(G) \}$$

and

$$\mathcal{H}_0(G) := \{A \in \mathcal{A}_\mu(G) : \ \mu(A + x) = 0, \ x \in G, \\ \text{for some probabilistic measure on } \mathcal{A}(G) \}$$

(Haar zero sets)

and

$$\mathcal{C}_0(G) := \{B \subset G \ : \ B \subset A \quad \text{for a set A} \in \mathcal{H}_0(\text{G}) \}$$

(Christensen zero sets) .

FACT. **In any Abelian Polish topological group the family $\mathcal{C}_0(G)$ of all Christensen zero sets yields a p.l.i σ-ideal.**

Theorem (F. C. Sánchez, 1999). **Let $(G, +)$ be an Abelian group with an invariant mean m and let $B \subset G$ be a set such that $m(G \setminus B) = 0$. Let further $\varrho : G \longrightarrow \mathbb{R}$ be a given function. Then, assuming that a function $f : G \longrightarrow \mathbb{R}$ satisfies any of the following three conditions:**

(i) $|f(x+y) - f(x) - f(y)| \leq \varrho(x)$
for all $x, y, x+y \in B$;

(ii) $|f(x+y) - f(x) - f(y)| \leq \varrho(x) + \varrho(y) - \varrho(x+y)$
for all $x, y, x+y \in B$;

(iii) $|f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i)| \leq \sum_{i=1}^{n} \varrho(x_i)$
for all $x_1, ...x_n,\ x_1 + \cdots x_n \in B$ **and** $n \in \mathbb{N}$,

there exists an additive function $a : G \longrightarrow \mathbb{R}$ such that

$$|f(x) - a(x)| \leq \varrho(x)$$

for all $x \in B$.

THEOREM. **Let** $(G,+)$ **be a group admitting an invariant mean** m **and let** $B \subset G$ **be a set such that** $m(G \setminus B) = 0$**. Let further** $(H,+)$ **be an Abelian group admitting sufficiently many real characters. If a map** $a : B \longrightarrow H$ **satisfies the condition**

$$x, y, x+y \in B \;\Rightarrow\; a(x+y) = a(x) + a(y)\,,$$

then there exists exactly one homomorphism $A : G \longrightarrow H$ **such that**

$$A(x) = a(x) \text{ for all } x \in B.$$

Theorem (F. C. Sánchez, 1999). **Let $(G,+)$ be an Abelian group admitting an invariant mean m and let $B \subset G$ be a set such that $m(G \setminus B) = 0$. Let further a real Banach space $(Y, \|\cdot\|)$ with a continuous projection π of its second dual onto Y and a function $\varrho : G \longrightarrow \mathbb{R}$ be given. Then, assuming that a function $f : G \longrightarrow Y$ satisfies any of the following two conditions:**

(i) $\|f(x+y) - f(x) - f(y)\| \leq \varrho(x)$
for all $x, y, x+y \in B$;

(ii) $\|f(x+y) - f(x) - f(y)\| \leq \varrho(x) + \varrho(y) - \varrho(x+y)$
for all $x, y, x+y \in B$,

then there exists an additive mapping $a : G \longrightarrow Y$ such that

$$\|f(x) - a(x)\| \leq 2\|\pi\|\varrho(x)$$

for all $x \in B$.

Theorem (F. C. Sánchez, 1999). **Let** $(G, +)$ **be an Abelian group admitting an invariant mean** m **and let** $B \subset G$ **be a set such that** $m(G \setminus B) = 0$. **Given two functions** $\alpha, \beta : B \longrightarrow \mathbb{R}$ **such that** $\alpha \leq \beta$ **and**

$$x, y, x + y \in B \Rightarrow \alpha(x + y) \geq \alpha(x) + \alpha(y),$$

as well as

$$x, y, x + y \in B \Rightarrow \beta(x + y) \leq \beta(x) + \beta(y),$$

there exists an additive functional $a : G \longrightarrow \mathbb{R}$ **such that**

$$\alpha(x) \leq a(x) \leq \beta(x) \textbf{ for all } x \in B.$$

Theorem. **Let X be a real linear topological space and let M be a mean on the space $\mathcal{B}(S,\mathbb{R})$, where S is a given nonempty set. Let further**

$$\mathcal{B}_M(S,X) := \left\{ f \in \mathcal{B}(S,X) : \bigvee_{\mathcal{M}(f)\in X} \bigwedge_{x^*\in X^*} x^*(\mathcal{M}(f)) = M(x^*\circ f) \right\}.$$

Then $\mathcal{B}_M(S,X)$ yields a linear subspace of the space $\mathcal{B}(S,X)$ and a mapping

$$\mathcal{B}_M(S,X) \ni f \longmapsto \mathcal{M}(f) \in X$$

is well defined, linear and satisfies the condition

$$\mathcal{M}(f) \in \operatorname{cl}\operatorname{conv} f(S), \quad f \in \mathcal{B}_M(S,X).$$

Theorem. **Let X stand for a real locally convex linear topological space and let M be a mean on the space $\mathcal{B}(S, \mathbb{R})$, where S is a given nonempty set. If a function $f : S \longrightarrow X$ enjoyes the property that**

$$\textbf{the set} \quad \textbf{cl conv } f(S) \quad \textbf{is weakly compact in } X\,,$$

then there exists exactly one element $\mathcal{M}(f) \in \mathrm{cl\,conv} f(S)$ such that

$$x^*(\mathcal{M}(f)) = M(x^* \circ f)$$

for all $x^* \in X^*$.

Proposition. **Let X stand for a real locally convex linear topological space and let M be a mean on the space $\mathcal{B}(S,\mathbb{R})$, where S is a given nonempty set. Then the set**

$$\mathcal{C}_{\mathcal{W}}(S,X) := \\ \{f : S \longrightarrow X : \operatorname{cl}\operatorname{conv} f(S) \quad \text{is weakly compact in } X\}$$

yields a linear subspace of the space $\mathcal{B}(S,X)$.

If, moreover, $(S,+)$ forms a semigroup, then that subspace is closed under translation in argument.

Theorem. **Let X stand for a real locally convex linear topological space and let $(S, +)$ be a semigroup admitting a left (right) invariant mean on the space $\mathcal{B}(S, \mathbb{R})$. The the linear space $\mathcal{C}_{\mathcal{W}}(S, X)$ is closed under left (right) translations in arument and admits a left (right) invariant mean, i.e. a linear operator. $\mathcal{M} : \mathcal{C}_{\mathcal{W}}(S, X) \longrightarrow X$ such that**

$$\mathcal{M}(f) \in \mathrm{cl\,conv} f(S)$$

for all $f \in \mathcal{C}_{\mathcal{W}}(S, X)$ and

$$\mathcal{M}({}_t f) = \mathcal{M}(f) \quad \text{resp.} \quad \mathcal{M}(f_t) = \mathcal{M}(f)$$

for all f $\in \mathcal{C}_{\mathcal{W}}(S, X)$ and $t \in S$.

Theorem. **Let X stand for a real locally convex linear topological space and let $(S,+)$ be a semigroup admitting a left invariant mean on the space $\mathcal{B}(S,\mathbb{R})$. Let further $F : S \longrightarrow 2^X \setminus \{\emptyset\}$ have the property that its values**

$$F(s) \text{ are convex and weakly compact for every } s \in S\,.$$

Then F admits an additive selection if and only if there exists a function $f : S \longrightarrow X$ such that

$$f(s+t) - f(t) \in F(s) \quad \text{for all} \quad s,t \in S\,.$$

Corollary. **Let** $(X, \|\cdot\|)$ **be a real reflexive Banach space and let** $(S, +)$ **be a semigroup admitting a left invariant mean on the space** $\mathcal{B}(S, \mathbb{R})$**. Let further** $\rho : S \longrightarrow [0, \infty)$ **and** $g : S \longrightarrow X$ **be given functions. Then a functional inequality**

$$\|f(s+t) - f(t) - g(s)\| \leq \rho(s)\,, \quad s, t \in S$$

admits a solution $f : S \longrightarrow X$ **and only if there exists an additive mapping** $A : S \longrightarrow X$**, such that**

$$\|g(s) - A(s)\| \leq \rho(s)\,, \quad s \in S.$$

Theorem. **Let $(X, \|\cdot\|)$ be a real reflexive Banach space and let $(G, +)$ be an amenable group. Given a number $\varepsilon \in [0, 1)$ and a function $f : G \longrightarrow X$ satisfying a functional inequality**

$$\|f(s+t) - f(s) - f(t)\| \leq \varepsilon \|f(s+t)\|\,, \quad s, t \in G,$$

there exist an additive map $A : G \longrightarrow X$ and an odd map $\varphi : A(G) \longrightarrow X$ such that

$$f = \varphi \circ A$$

and

$$\|\varphi(u) - \varphi(v) - (u - v)\| \leq \frac{2\varepsilon}{1-\varepsilon} \|u - v\|\,, \quad u, v \in A(G).$$

Conversely, for each additive map $A : G \longrightarrow X$ and each odd map $\varphi : A(G) \longrightarrow X$ satisfying the inequality

$$\|\varphi(u) - \varphi(v) - (u - v)\| \leq \delta \|u - v\|\,, \quad u, v \in A(G)$$

with some $\delta \in [0, 1)$, the function $f := \varphi \circ A$ yields a solution to the functional inequality

$$\|f(s+t) - f(s) - f(t)\| \leq \frac{2\delta}{1-\delta} \|f(s+t)\|\,, \quad s, t \in G.$$

Theorem. **Let $(X, \|\cdot\|)$ be a real reflexive Banach space and let $(G, +)$ be an amenable group. Given a number $\varepsilon \in [0, \frac{1}{2})$ and a function $f : G \longrightarrow X$ satisfying a functional inequality**

$$\|f(s+t) - f(s) - f(t)\| \leq \varepsilon \|f(s) + f(t)\|, \quad s, t \in G,$$

there exist an additive map $A : G \longrightarrow X$ and an odd map $\varphi : A(G) \longrightarrow X$ such that

$$f = \varphi \circ A$$

and

$$\|\varphi(u) - \varphi(v) - (u - v)\| \leq \frac{2\varepsilon}{1 - 2\varepsilon} \|u - v\|, \quad u, v \in A(G).$$

Conversely, for each additive map $A : G \longrightarrow X$ and each odd map $\varphi : A(G) \longrightarrow X$ satisfying the inequality

$$\|\varphi(u) - \varphi(v) - (u - v)\| \leq \delta \|u - v\|, \quad u, v \in A(G)$$

with some $\delta \in [0, 1)$, the function $f := \varphi \circ A$ yields a solution to the functional inequality

$$\|f(s+t) - f(s) - f(t)\| \leq \frac{2\delta}{1 - \delta} \|f(s) + f(t)\|, \quad s, t \in G.$$

Let $L^1(\mathbb{R})$ denote the algebra of all complex Lebesgue integrable functions on $\mathbb{R}$ (ℓ_1-almost everywhere equal functions being identified) with convolution multiplication

$$(x * y)(t) := \int_{\mathbb{R}} x(t-s)y(s)\, d\,\ell_1(s), \quad x, y \in L^1(\mathbb{R}),$$

and with integral norm

$$\|x\| := \int_{\mathbb{R}} |x(t)|\, d\,\ell_1(t), \quad x, y \in L^1(\mathbb{R}).$$

Looking for an analytic form of linear multiplicative functionals on $L^1(\mathbb{R})$, we are faced to the problem of finding solutions of the Cauchy type functional equation

$$F(x * y) = F(x)F(y), \quad x, y \in L^1(\mathbb{R}),$$

in the class of functionals $F \in L^1(\mathbb{R})^*$. Since the dual space $L^1(\mathbb{R})^*$ is isometrically isomorphic with the space $L^\infty(\mathbb{R})$, there exists exactly one function $f \in L^\infty(\mathbb{R})$ such that

$$F(x) = \int_{\mathbb{R}} x(t)f(t)\, d\,\ell_1(t), \quad x \in L^1(\mathbb{R}).$$

Fubini's theorem jointly with Stone-Weierstrass approximation theorem allow to obtain the following relationship

$$\int_{\mathbb{R}^2} \varphi(s,t)\,[f(s+t) - f(s)f(t)]\; d\,\ell_2(s,t),$$

that is valid for all functions φ from the algebra $L^1(\mathbb{R}^2)$. This forces f to satisfy the exponential Cauchy functional equation

$$f(s+t) = f(s)f(t)$$

for ℓ_2-almost all pairs $(s,t) \in \mathbb{R}^2$.

Now, applying (some modification) of N. G. de Bruijn's result (1966), we infer that there exists exactly one function $g : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$g(s+t) = g(s)g(t) \quad \textbf{for all pairs} \quad (s,t) \in \mathbb{R}^2$$

and

$$f(t) = g(t) \quad \textbf{for } \ell_1 - \textbf{almost all} \quad t \in \mathbb{R}\,.$$

Consequently, one has

$$F(x) = \int_{\mathbb{R}} x(t)e^{ipt}d\,\ell_1(t)\,, \qquad x \in L^1(\mathbb{R})$$

for some $p \in \mathbb{R}$.

Therefore any linear multiplicative functional F on the convolution algebra $L^1(\mathbb{R})$ is nothing else but the Fourier transform.

THEOREM **(L. Molnar & M. Györy, 1998). Given a Hausdorff topological space S with first axiom of countability let $C(S,\mathbb{C})$ denote the Banach algebra of all continuous complex functions on S, with the uniform convergence norm. Assume that $\Phi : C(S,\mathbb{C}) \longrightarrow C(S,\mathbb{C})$ is a linear bijection preserving the diameters of ranges of the members of the algebra $C(S,\mathbb{C})$, i.e.**

$$\operatorname{diam}\Phi(f)(S) = \operatorname{diam} f(S), \quad f \in C(S,\mathbb{C}).$$

Then there exists a number $\alpha \in \mathbf{C}\backslash\{0\}$, a homeomorphism $\varphi : S \longrightarrow S$ and a linear functional $x : C(S,\mathbb{C}) \longrightarrow \mathbb{C}$ such that

$$\Phi(f) = \alpha \cdot f \circ \varphi + x(f) \cdot \chi_S$$

for all $f \in C(S,\mathbb{C})$.

THEOREM **(L. Molnar, 2002). Each bijective solution Φ of the functional equation**

$$\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$$

on a standard algebra of operators over the linear space of dimension at least 2 is automatically additive, i.e. Φ yields a Jordan isomorphism.

Theorem **(K. Baron, 2005). Given a complex Hilbert space** $(X, (\cdot|\cdot))$ **let** $T : X \longrightarrow X$ **be a positive self-adjoint operator. Then** T **is embeddable in a semigroup of operators, i.e. there exists a map** $F : X \times (0, \infty) \longrightarrow X$ **such that**

$$F(x, t+s) = F\left(F(x, s), t\right)$$

for all $x \in X,\ s, t \in (0, \infty)$ **and**

$$F(\cdot, 1) = T\,.$$

Hahn-Banach type theorems

Theorem ("algebraic" version of the classical Hahn-Banach theorem). **Let X_0 be a linear subspace of a real linear space X and let $p : X \longrightarrow \mathbb{R}$ be a sublinear (=subadditive and positive homogeneous) functional. Then for every linear functional $f_0 : X_0 \longrightarrow \mathbb{R}$ dominated by p, i.e.**

$$f_0(x) \leq p(x) \quad \text{for all} \quad x \in X_0\,,$$

there exists a linear functional $f : X_0 \longrightarrow \mathbb{R}$ such that

$$f(x) = f_0(x) \quad \text{for} \quad x \in X_0$$

and

$$f(x) \leq p(x) \quad \text{for all} \quad x \in X\,.$$

Abstract version: (Rodé's theorem)

DEFINITION. **Let X be a nonempty set and let n, m be positive integers. We say that the maps $\sigma : X^m \longrightarrow X$ and $\tau : X^n \longrightarrow X$ commute if and only if**

$$\sigma\left(\tau(x_{1,1}, ..., x_{1,n}), ..., \tau(x_{m,1}, ..., x_{m,n})\right)$$
$$= \tau\left(\sigma(x_{1,1}, ..., x_{m,1}), ..., \sigma(x_{1,n}, ..., x_{m,n})\right)$$

for all elements $x_{i,j}$, $i = 1, ..., m$, $j = 1, ..., n$, from the set X.

Theorem **(G. Rodé , 1978). Let X be a nonempty set and let Γ stand for a family of pairwise commuting mappings. Let further $f, g : X \longrightarrow [-\infty, \infty)$ be two functions such that $g \leq f$. Then there exists a function $\varphi : X \longrightarrow [-\infty, \infty)$ between g and f with the following property: if for some $\sigma : X^n \longrightarrow X$ from the family Γ and for some nonnegative scalars $\alpha_1, ..., \alpha_n$ the inequalities**

$$f(\sigma(x_1, ..., x_n)) \leq \sum_{k=1}^{n} \alpha_k f(x_k)\,,$$

and

$$g(\sigma(x_1, ..., x_n)) \geq \sum_{k=1}^{n} \alpha_k g(x_k)\,,$$

are valid for all $x_1, ..., x_n$ from X, then

$$\varphi(\sigma(x_1, ..., x_n)) = \sum_{k=1}^{n} \alpha_k \varphi(x_k)$$

for all $x_1, ..., x_n \in X$.

Definition. **We say that a group $(G, +)$ belongs to the class $\mathcal{G}$ if and only if for each subadditive functional $p : G \longrightarrow \mathbb{R}$ there exists an additive functional $\mathbf{G} \longrightarrow \mathbb{R}$ such that $a \leq p$.**

Theorem **(R. Badora, 2006). Let $(G, +)$ be a group. Then $(G, +) \in \mathcal{G}$ if and only if for each subgroup $(G_0, +)$ of the group $(G, +)$ and for every subadditive functional $p : G \longrightarrow \mathbb{R}$ such that**

$$M(x) := \sup\{p(-a + x + a) - p(x) : \, a \in G_0\} \in \mathbb{R}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} M(nx) = 0 \,,$$

for all $x \in G$, every additive functional $a_0 : G_0 \longrightarrow \mathbb{R}$ such that $a_0 \leq p|_{G_0}$, admits an extension to an additive functional $a : G \longrightarrow \mathbb{R}$ such that $a \leq p$.

COROLLARY. **Let a group** $(G,+)$ **be a member of the class** $\mathcal{G}$ **and let** $p : G \longrightarrow \mathbb{R}$ **be a subadditive functional such that**

$$p(2x) = 2p(x), \qquad x \in G\,.$$

Then for each subgroup $(G_0,+)$ **of the group** $(G,+)$ **and for every additive functional** $a_0 : G_0 \longrightarrow \mathbb{R}$ **such that** $a_0 \leq p|_{G_0}$**, there exists its additive extension** $a : G \longrightarrow \mathbb{R}$ **such that** $a \leq p$.

$$\boldsymbol{Abel} \subset \boldsymbol{Amen} \subset \mathcal{G} \subset \boldsymbol{Hyers}$$

$$\bigcup$$

$$\boldsymbol{weakly\ commutative}$$

Theorem. **Let a group** $(G,+)$ **be a member of the class** $\mathcal{G}$ **and let** $(Y,\|\cdot\|_Y)$ **be a real normed linear space. Let further** $f: X \longrightarrow Y$ **be a solution to the functional equation**

$$\text{(FM)} \qquad \|f(x+y)\|_Y = \|f(x)+f(y)\|_Y\,, \quad x,y \in X\,.$$

Then there exist: a nonempty set $T \subset \mathbb{R}^X$**, an additive operator** $A : X \longrightarrow B(T,\mathbb{R})$ **and an odd isometry** $I : A(X) \longrightarrow Y$ **such that**

$$f(x) = I(A(x))\,, \quad x \in X\,.$$

Conversely, for every real normed linear space $(Z,\|\cdot\|_Z)$**, for each additive operator** $A : X \longrightarrow Z$ **and for any odd isometry** $I : A(X) \longrightarrow Y$ **the superposition** $f := I \circ A$ **yields a solution to equation** (FM)**.**

Theorem. **Let** $(X, \|\cdot\|_X)$ **and** $(Y, \|\cdot\|_Y)$ **be real normed linear spaces. Let further** $f : X \longrightarrow Y$ **be a solution to the functional equation** (FM)**, such that the functional** $p : X \longrightarrow \mathbb{R}$ **given by the formula**

$$p(x) := \|f(x)\|_Y \,, \quad x \in X \,,$$

satisfies any regularity condition forcing the continuity of Jensen convex functionals. Then there exist: a nonempty set $T \subset \mathbb{R}^X$**, a continuous linear operator** $L : X \longrightarrow B(T, \mathbb{R})$ **and an odd isometry** $I : L(X) \longrightarrow Y$ **such that**

$$f(x) = I(L(x)) \,, \quad x \in X \,.$$

Conversely, for every real normed linear space $(Z, \|\cdot\|_Z)$**, for each continuous linear operator** $L : X \longrightarrow Z$ **and for any odd isometry** $I : L(X) \longrightarrow Y$ **the superposition** $f := I \circ L$ **yields a solution to equation** (FM) **with a continuous functional** p**.**

Theorem. **Any solution of the functional equation** (FM) **mapping a group from the class** $\mathcal{G}$ **into a strictly convex real normed linear space, is automatically additive.**

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