Minimal projections onto hyperplanes in vector-valued sequence spaces

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Definition

Let X be a Banach space and V be a linear, closed subspace of X. Then by $\mathcal{P}(X, V)$ we denote the set of all linear projections continuous with respect to the operator norm. An operator $P : X \to V$ is called a projection, if $P|_V = id_V$. A projection $P_0 \in \mathcal{P}(X, V)$ is called minimal if

$$\|P_0\| = \lambda(V, X) := \inf\{\|P\| : P \in \mathcal{P}(X, V)\}.$$

For all $x \in X$

$$\|x - Px\| \leq \|Id - P\| \cdot \mathrm{dist}(x, V) \leq (1 + \|P\|) \cdot \mathrm{dist}(x, V),$$

where $\mathrm{dist}(x, V) := \inf\{\|x - v\| : v \in V\}$.

Definition

Let $\{(X_n, \ \|\cdot\|^{(n)})\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We define

$$c_0(\{X_n\}_{n\in\mathbb{N}}) := \{\{x_n\}_{n\in\mathbb{N}} : \ x_n \in X_n \quad \text{and} \quad \lim_{n\to\infty} \|x_n\|^{(n)} = 0\};$$

$$l_1(\{X_n\}_{n\in\mathbb{N}}) := \{\{x_n\}_{n\in\mathbb{N}} : \ x_n \in X_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|x_n\|^{(n)} < \infty\}.$$

These sequence spaces are Banach spaces when equipped with the following norms:

$$\|x\| = \sup_{n\in\mathbb{N}} \|x_n\|^{(n)} \quad \text{for} \quad x \in c_0(\{X_n\}_{n\in\mathbb{N}});$$

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\|^{(n)} \quad \text{for} \quad x \in l_1(\{X_n\}_{n\in\mathbb{N}}).$$

Theorem

Let $\left\{\left(X_n, \ \|\cdot\|^{(n)}\right)\right\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We have

(i) $c_0(\{X_n\}_{n\in\mathbb{N}})^* = l_1(\{X_n^*\}_{n\in\mathbb{N}})$ in the sense that for every $f \in c_0(\{X_n\}_{n\in\mathbb{N}})^*$ there is a unique $\{f_n\}_{n\in\mathbb{N}} \in l_1(\{X_n^*\}_{n\in\mathbb{N}})$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x_n) \quad \textit{for every} \quad x \in c_0(\{X_n\}_{n\in\mathbb{N}})$$

and the map $f \mapsto \{f_n\}_{n\in\mathbb{N}}$ is a linear isometry of $c_0(\{X_n\}_{n\in\mathbb{N}})^*$ onto $l_1(\{X_n^*\}_{n\in\mathbb{N}})$.

(ii) $l_1(\{X_n\}_{n\in\mathbb{N}})^* = l_\infty(\{X_n^*\}_{n\in\mathbb{N}})$ in the sense that for every $f \in l_1(\{X_n\}_{n\in\mathbb{N}})^*$ there is a unique $\{f_n\}_{n\in\mathbb{N}} \in l_\infty(\{X_n^*\}_{n\in\mathbb{N}})$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x_n) \quad \textit{for every} \quad x \in l_1(\{X_n\}_{n\in\mathbb{N}})$$

and the map $f \mapsto \{f_n\}_{n\in\mathbb{N}}$ is a linear isometry of $l_1(\{X_n\}_{n\in\mathbb{N}})^*$ onto $l_\infty(\{X_n^*\}_{n\in\mathbb{N}})$.

Lemma

Let X be a normed linear space and let f be a continuous nonzero linear functional on X. Each projection of X onto the hyperplane $\ker f$ is of the following form:

$$P_w(x) = x - f(x)w \qquad \textit{for all } x \in X,$$

where $w \in X$ and $f(w) = 1$.

Lemma

Let $\left\{ \left(X_n, \ \| \cdot \|^{(n)} \right) \right\}_{n \in \mathbb{N}}$ be a sequence of Banach spaces and let $f \in S(l_1(\{X_n^*\}_{n \in \mathbb{N}}))$ and $V := \ker f$. If $P_w \in \mathcal{P}(c_0(\{X_n\}_{n \in \mathbb{N}}), V)$, then

$$\|P_w\| \geq \max_{i \in \mathbb{N}} \{|1 - f_i(w_i)| + (1 - \|f_i\|^{(i)})\|w_i\|^{(i)}\}.$$

Theorem

Let $\{X_n\}_{n\in\mathbb{N}}, f, V$ be as before. If $\|f\|_\infty = \sup_{i\in\mathbb{N}} \|f_i\|^{(i)} < \frac{1}{2}$, then

$$\lambda(V, c_0(\{X_n\}_{n\in\mathbb{N}})) \geq 1 + \left(\sum_{i=1}^{\infty} \frac{\|f_i\|^{(i)}}{1 - 2\|f_i\|^{(i)}} \right)^{-1} \quad (1)$$

It is worth mentioning that in the classical case of the space c_0 we have equality in (1) (see [1]). The next theorem shows that in general case the equality does not hold.

Theorem

Let $X = \mathcal{C}([0,1])$ and $V = \ker f$, where $f \in S(X^*)$. Then $\lambda(V, c_0(X)) = 2$.

[1]. J. Blatter, E.W. Cheney, *Minimal projections onto hyperplanes in sequence spaces,* Ann. Mat. Pura Appl Vol. 101 (1974), 215-227.

Theorem

Let $\left\{\left(X_n, \ \|\cdot\|^{(n)}\right)\right\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces and let $f \in S(l_1(\{X_n^*\}_{n\in\mathbb{N}}))$ and $V := \ker f$. If $\|f\|_\infty \geq \frac{1}{2}$, then there is a projection of norm one in $\mathcal{P}(c_0(\{X_n\}_{n\in\mathbb{N}}), V)$ iff there exists $w_i \in X_i$ such that

1. $f_i(w_i) = 1$ and $\|f_i\|^{(i)} \geq \frac{1}{2}$;
2. $\|Id - f_i(\cdot)w_i\| = 1$;
3. $\|v - \lambda w_i\|^{(i)} \leq 1$ for all $v \in B(\ker f_i)$ and for all $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1 - \|f_i\|^{(i)}$.

Corollary

Let $(X_n, \ \|\cdot\|^{(n)})$ be a strictly convex Banach space and let $\dim X_n > 1$ for any $n \in \mathbb{N}$. Let $f \in l_1(\{X_n^*\}_{n\in\mathbb{N}})$. Then there exists a projection of norm 1 in $\mathcal{P}(c_0(\{X_n\}_{n\in\mathbb{N}}), \ker f)$ iff there is exactly one $i \in \mathbb{N}$ such that $f_i \neq 0$ and for this i there exists $\widetilde{P}_{w_i} \in \mathcal{P}(X_i, \ker f_i)$ such that $\|\widetilde{P}_{w_i}\| = 1$.

Theorem

Let $X = l_1^n$ for $2 < n \leq \infty$. Let $f \in S(l_1(l_\infty^n))$ and $V = \ker f$. Then there is a projection of norm one in $\mathcal{P}(c_0(X), V)$ iff
I. there is exactly one $i \in \mathbb{N}$ such that $f_i \neq 0$;
II. f_i has at most two nonzero coordinates.

Theorem

Let $X = l_1^2$, $f \in S(l_1(l_\infty^2))$ and $V = \ker f$. Then there is a projection of norm one in $\mathcal{P}(c_0(X), V)$ iff there exists $i \in \mathbb{N}$ such that $\|f_i\|_1 \geq 1$.

Lemma

Let X_n be a sequence of Banach spaces and let $f \in l_\infty(\{X_n^*\}_{n\in\mathbb{N}})$ and $V := \ker f$. Let $P_w \in \mathcal{P}(l_1(\{X_n\}_{n\in\mathbb{N}}), V)$, then

$$\|P_w\| \geq \max_{i\in\mathbb{N}}\{|1 - \|f_i\|^{(i)}\|w_i\|^{(i)}| + (\|w\| - \|w_i\|^{(i)}\|f_i\|^{(i)}\}.$$

Theorem

Let $\{X_n\}_{n\in\mathbb{N}}, f, V$ be the same as before . Then

$$\lambda(l_1(\{X_n\}_{n\in\mathbb{N}}), V) \geq \lambda(l_1, \widetilde{V}), \tag{2}$$

where $\widetilde{V} = \ker(\|f_1\|^{(1)}, \|f_2\|^{(2)}, \dots)$.

Theorem

Let $X := \mathcal{C}([0,1])$, $f \in S(l_\infty(X^*))$ and $V = \ker f$. Then $\lambda(V, l_1(X)) = 2$.

Theorem

Let X_n be a sequence of Banach spaces and let $f \in l_\infty(\{X_n^*\}_{n\in\mathbb{N}})$ and $V := \ker f$. If there exists a projection $P \in \mathcal{P}(l_1(\{X_n\}_{n\in\mathbb{N}}), V)$ of norm 1 then at most two coordinates of f are different from zero.

Example

Let $X := l_\infty^2$ and $f \in l_\infty(l_1^2)$ such that $f_1 = f_2 = (1, 0)$ and $f_j = 0$ for $j \in \{3, 4, \dots\}$. Then $\lambda(\ker f, l_1(l_\infty^2)) > 1$.

For more details, see: B. Deregowska, B. Lewandowska, *Minimal projections onto hyperplanes in vector-valued sequence spaces,* J. Approximation Theory Vol. 194 (2015), 1-13.

Thank you for your attention