# Some Classes of Operators on Separable Banach Spaces

Richard Becker

Institut de Mathématiques de Jussieu
& Université Pierre-et-Marie Curie (Paris-06)

Function Spaces XI
Zielona Gora
July 6-10, 2015

# Outline

The origin of my talk is the following natural question: Given two Banach spaces $B$ and $E$, is it possible to describe all the operators from $B$ to $E$ that can be extended to every Banach space containing $B$.
In this talk we shall deal with the case where $B$ is contained in the Banach space $\mathcal{C}([0,1])$, denoted by $\mathcal{C}$ for short.
$B$ will be always assumed to be separable. It is a well known fact that there exists a linear map $\phi : B \to \mathcal{C}$ such that, for every $x \in B$: $\|\phi(x)\| = \|x\|$.

We shall place ourselves within the framework of the following definition.

### Definition

Let $1 \leq C < \infty$. We denote by $\mathcal{F}(B, E, C)$ the set of all operators $T : B \to E$ such that there exists a linear map $\phi : B \to \mathcal{C}$, with $\|x\| \leq \|\phi(x)\| \leq C\|x\|$, for every $x \in B$, and there exists $T_\phi : \mathcal{C} \to E$ satisfying $T = T_\phi \circ \phi$ on $B$.
If $T \in \mathcal{F}(B, E, C)$ we set $\|T\|_C = \inf\{\|T_\phi\|\}$, where the inf is taken over all $\phi$ and all $T_\phi$ as above.
We denote by $\mathcal{F}(B, E, \infty)$ the union of all the spaces $\mathcal{F}(B, E, C)$.

## Theorem

*1)* $\mathcal{F}(B,E,C)$ *is a normed space when it is equipped with* $\|T\|_C$.
*2) For each* $T \in \mathcal{F}(B,E,C)$ *then* $\|T\| \leq C\|T\|_C$.
*3) For* $1 \leq C_1 \leq C_2 < \infty$ *then* $\mathcal{F}(B,E,C_1) \subset \mathcal{F}(B,E,C_2)$ *and, if* $T \in \mathcal{F}(B,E,C_1)$, *then* $C_2\|T\|_{C_2} \leq C_1\|T\|_{C_1}$.
*4)* $\mathcal{F}(B,E,\infty)$ *is a normed space when it is equipped with* $\|T\|_\infty = \lim C\|T\|_C$ *as* $C \to \infty$, *and* $\|T\| \leq \|T\|_\infty$.
*The Banach spaces associated are denoted by* $\overline{\mathcal{F}}(B,E,C)$ *for* $1 \leq C \leq \infty$ *and we keep the notation* $\| \ \|_C$.

Some cases are immediate:
1) When $B$ can be embedded in $\mathcal{C}$ as a complemented subspace of $\mathcal{C}$. This is the case when $B$ is a space $\mathcal{C}(K)$, where $K$ is an infinite compact metric space, by Milutin's Theorem and Milutin's Lemma.
In this cases, for every Banach space $E$, every $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$ for any $C$.
2) When $E$ is an $L^\infty$ space, associated with a $\sigma$-finite measure, or when $E$ is the space $c_0$.
In this cases (by a Theorem of Zippin for $c_0$), for every Banach space $B$, every $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$ for any $C$.

## Proposition

*1) Every element of* $\overline{\mathcal{F}}(B, E, C)$ *can be viewed as a continuous operator from* $B$ *to* $E$*, for* $1 \leq C \leq \infty$.
*2) If* $T \in \mathcal{F}(B, E, C)$ *and* $V \in \mathcal{L}(E)$ *then* $V \circ T \in \mathcal{F}(B, E, C)$ *and* $\|V \circ T\|_C \leq \|V\| \|T\|_C$.
*3) If* $E = B$ *the space* $\overline{\mathcal{F}}(B, B, C)$ *is an algebra, for every* $1 \leq C \leq \infty$.
*When* $1 \leq C < \infty$ *the map* $T \to C\|T\|_C$ *satisfies:*

$$C\|T_1 \circ T_2\|_C \leq (C\|T_1\|_C)(C\|T_2\|_C)$$

*for* $T_1, T_2 \in \overline{\mathcal{F}}(B, B, C)$.
*Moreover* $\|T_1 \circ T_2\|_\infty \leq \|T_1\|_\infty \|T_2\|_\infty$ *for* $T_1, T_2 \in \overline{\mathcal{F}}(B, B, \infty)$.

The following Proposition gives some informations concerning the role of the constant $C$ in the definition of $\mathcal{F}(B, E, C)$.

## Proposition

*Let* $1 \leq C \leq \infty$. *If* $U \in \mathcal{L}(B)$ *satisfies* $m\|x\| \leq \|U(x)\| \leq M\|x\|$ *for every* $x \in B$, *with* $0 < m \leq M < \infty$, *then, for every* $T \in \mathcal{F}(B, E, C)$:
$T \circ U \in \mathcal{F}(B, E, CM/m)$ *and* $\|T \circ U\|_{CM/m} \leq m\|T\|_C$.
*Moreover* $m\|T\|_\infty \leq \|T \circ U\|_\infty \leq M\|T\|_\infty$.

This result shows that the space $\mathcal{F}(B, E, \infty)$ does not change if $B$ is replaced by one of its isomorphic copy.

The following Proposition gives some informations concerning the role of the constant $C$ in the definition of $\mathcal{F}(B, E, C)$.

## Proposition

*Let* $1 \leq C \leq \infty$. *If* $U \in \mathcal{L}(B)$ *satisfies* $m\|x\| \leq \|U(x)\| \leq M\|x\|$ *for every* $x \in B$, *with* $0 < m \leq M < \infty$, *then, for every* $T \in \mathcal{F}(B, E, C)$*:*
$T \circ U \in \mathcal{F}(B, E, CM/m)$ *and* $\|T \circ U\|_{CM/m} \leq m\|T\|_C$.
*Moreover* $m\|T\|_\infty \leq \|T \circ U\|_\infty \leq M\|T\|_\infty$.

This result shows that the space $\mathcal{F}(B, E, \infty)$ does not change if $B$ is replaced by one of its isomorphic copy.

## Proposition

*If $E$ does not contain a copy of $c_0$ then every element of $\overline{\mathcal{F}}(B, E, C)$ is weakly compact for $1 \leq C \leq \infty$.*

## Proposition

*If $E$ does not contain a copy of $\mathcal{C}$ then, for every $T \in \overline{\mathcal{F}}(B, E, C)$ and $1 \leq C \leq \infty$, the space $T'(E')$ is separable.*

## Proposition

*If $E'$ is an $L^1$ space then every compact operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\|_1 = \|T\|$.*

### Proposition

*If $E$ does not contain a copy of $c_0$ then every element of $\overline{\mathcal{F}}(B, E, C)$ is weakly compact for $1 \leq C \leq \infty$.*

### Proposition

*If $E$ does not contain a copy of $\mathcal{C}$ then, for every $T \in \overline{\mathcal{F}}(B, E, C)$ and $1 \leq C \leq \infty$, the space $T'(E')$ is separable.*

### Proposition

*If $E'$ is an $L^1$ space then every compact operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\|_1 = \|T\|$.*

### Proposition

*If $E$ does not contain a copy of $c_0$ then every element of $\overline{\mathcal{F}}(B, E, C)$ is weakly compact for $1 \leq C \leq \infty$.*

### Proposition

*If $E$ does not contain a copy of $\mathcal{C}$ then, for every $T \in \overline{\mathcal{F}}(B, E, C)$ and $1 \leq C \leq \infty$, the space $T'(E')$ is separable.*

### Proposition

*If $E'$ is an $L^1$ space then every compact operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\|_1 = \|T\|$.*

**Proposition**

*If $B$ is reflexive then all the elements of $\overline{\mathcal{F}}(B, E, C)$ are compact operators, for $1 \leq C \leq \infty$.*

**Proposition**

*Let $B$ be a subspace of $c_0$ and $E = \mathcal{C}(K)$ where $K$ is a compact Hausdorff space. Then every operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\| \leq \|T\|_1 \leq 2\|T\|$.*

## Proposition

*If $B$ is reflexive then all the elements of $\overline{\mathcal{F}}(B, E, C)$ are compact operators, for $1 \leq C \leq \infty$.*

## Proposition

*Let $B$ be a subspace of $c_0$ and $E = \mathcal{C}(K)$ where $K$ is a compact Hausdorff space. Then every operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\| \leq \|T\|_1 \leq 2\|T\|$.*

## Definition

Let $X, Y$ be two Banach spaces, and $1 \leq p < \infty$. A linear operator $T : X \to Y$ is said to be *p-summing* if, for every finite sequence $x_1 \dots x_n$ in $X$, one has:

$$(\sum_1^n ||T(x_i)||^p)^{1/p} \leq c.\sup\{(\sum_1^n |\langle x', x_i\rangle|^p)^{1/p} : x' \in X', \|x'\| = 1\}$$

where $c$ is some constant.
The smallest possible constant is denoted by $\pi_p(T)$.
We denote by $\pi_p(X, Y)$ the space of all this operators.

## Proposition

*Let $T \in \pi_2(B, E)$. Then, for every $1 \leq C \leq \infty$, $T \in \mathcal{F}(B, E, C)$, and $\|T\|_\infty \leq \pi_2(T)$, and $C\|T\|_C \leq \pi_2(T)$ for $1 \leq C < \infty$.*

## Definition

Let $X, Y$ be two Banach spaces, and $1 \leq p < \infty$. A linear operator $T : X \to Y$ is said to be *p-summing* if, for every finite sequence $x_1 \dots x_n$ in $X$, one has:

$$(\sum_1^n ||T(x_i)||^p)^{1/p} \leq c. \sup\{(\sum_1^n |\langle x', x_i \rangle|^p)^{1/p} : x' \in X', \|x'\| = 1\}$$

where $c$ is some constant.
The smallest possible constant is denoted by $\pi_p(T)$.
We denote by $\pi_p(X, Y)$ the space of all this operators.

## Proposition

*Let $T \in \pi_2(B, E)$. Then, for every $1 \leq C \leq \infty$, $T \in \mathcal{F}(B, E, C)$, and $\|T\|_\infty \leq \pi_2(T)$, and $C\|T\|_C \leq \pi_2(T)$ for $1 \leq C < \infty$.*

## Definition

Let $X$ be a Banach space and $2 \leq q < \infty$. $X$ is said to be of *cotype-$q$* if, for every finite sequence $x_1 \dots x_n$, we have:

$$(\sum_1^n \|x_i\|^q)^{1/q} \leq c.(\int \| \sum_1^n r_i(u) x_i \|^q du)^{1/q}$$

where $c$ is some constant and $(r_i)$ is the sequence of Rademacher variables. Let $C_q(X)$ denote the best constant $c$.

## Proposition

*If $E$ has cotype 2 then, for every $1 \leq C \leq \infty$, the space $\overline{\mathcal{F}}(B, E, C)$ is nothing but the space of all 2-summing operators $T : B \to E$, and $C\|T\|_C \leq \pi_2(T) \leq AC\|T\|_C$, where $A$ is some constant, for $1 \leq C \leq \infty$, and $\|T\|_\infty \leq \pi_2(T) \leq A\|T\|_\infty$.*

## Proposition

*If $E$ is of finite cotype $q$ then every $T \in \overline{\mathcal{F}}(B, E, C)$ is $r$-summing for every $r > q$ and for $1 \leq C \leq \infty$. Then $\pi_r(T) \leq A_r C\|T\|_C$, where $A_r$ is some constant, for $1 \leq C \leq \infty$ and $\pi_r(T) \leq A_r\|T\|_\infty$.*
*In particular, every $T \in \overline{\mathcal{F}}(B, E, C)$ is weakly compact for every $1 \leq C \leq \infty$.*

A Banach space $E$, with no cotype, can be such that $c_0 \not\subset E$.

## Proposition

*If $E$ has cotype 2 then, for every $1 \leq C \leq \infty$, the space $\overline{\mathcal{F}}(B, E, C)$ is nothing but the space of all 2-summing operators $T : B \to E$, and $C\|T\|_C \leq \pi_2(T) \leq AC\|T\|_C$, where $A$ is some constant, for $1 \leq C \leq \infty$, and $\|T\|_\infty \leq \pi_2(T) \leq A\|T\|_\infty$.*

## Proposition

*If $E$ is of finite cotype $q$ then every $T \in \overline{\mathcal{F}}(B, E, C)$ is $r$-summing for every $r > q$ and for $1 \leq C \leq \infty$. Then $\pi_r(T) \leq A_r C\|T\|_C$, where $A_r$ is some constant, for $1 \leq C \leq \infty$ and $\pi_r(T) \leq A_r\|T\|_\infty$.*
*In particular, every $T \in \overline{\mathcal{F}}(B, E, C)$ is weakly compact for every $1 \leq C \leq \infty$.*

A Banach space $E$, with no cotype, can be such that $c_0 \not\subset E$.

## Proposition

*If $E$ has cotype 2 then, for every $1 \leq C \leq \infty$, the space $\overline{\mathcal{F}}(B, E, C)$ is nothing but the space of all 2-summing operators $T : B \to E$, and $C\|T\|_C \leq \pi_2(T) \leq AC\|T\|_C$, where $A$ is some constant, for $1 \leq C \leq \infty$, and $\|T\|_\infty \leq \pi_2(T) \leq A\|T\|_\infty$.*

## Proposition

*If $E$ is of finite cotype $q$ then every $T \in \overline{\mathcal{F}}(B, E, C)$ is $r$-summing for every $r > q$ and for $1 \leq C \leq \infty$. Then $\pi_r(T) \leq A_r C\|T\|_C$, where $A_r$ is some constant, for $1 \leq C \leq \infty$ and $\pi_r(T) \leq A_r\|T\|_\infty$.*
*In particular, every $T \in \overline{\mathcal{F}}(B, E, C)$ is weakly compact for every $1 \leq C \leq \infty$.*

A Banach space $E$, with no cotype, can be such that $c_0 \not\subset E$.

Here is a stronger version of our Definition.

### Definition

Let $1 \leq C < \infty$. We denote by $\mathcal{G}(B, E, C)$ the set of all operators $T : B \to E$ such that, for every linear map $\phi : B \to \mathcal{C}$ with $\|x\| \leq \phi(x) \leq C\|x\|$, for every $x \in B$, there exists $T_\phi : \mathcal{C} \to E$ with $T = T_\phi \circ \phi$ on $B$ and such that $\sup\{\|T_\phi\|\} < \infty$, where the sup is taken on all such $\phi$.
We denote by $|||T|||_C$ the smallest number $K$ such that $\|T_\phi\| \leq K$ for all such $\phi$.

(There exits a linear map $\phi \ldots$) is replaced by (For every linear map $\phi \ldots$)

Here is a stronger version of our Definition.

### Definition

Let $1 \leq C < \infty$. We denote by $\mathcal{G}(B, E, C)$ the set of all operators $T : B \to E$ such that, for every linear map $\phi : B \to \mathcal{C}$ with $\|x\| \leq \phi(x) \leq C\|x\|$, for every $x \in B$, there exists $T_\phi : \mathcal{C} \to E$ with $T = T_\phi \circ \phi$ on $B$ and such that $\sup\{\|T_\phi\|\} < \infty$, where the sup is taken on all such $\phi$.
We denote by $|||T|||_C$ the smallest number $K$ such that $\|T_\phi\| \leq K$ for all such $\phi$.

(There exits a linear map $\phi \ldots$) is replaced by (For every linear map $\phi \ldots$)

1) If $C_1 \leq C_2$ then $\mathcal{G}(B, E, C_2) \subset \mathcal{G}(B, E, C_1)$ (the order is reverse).
2) $\mathcal{G}(B, E, 1) \subset \mathcal{F}(B, E, 1)$.
3) These spaces contains the space of 2-summing operators.
4) When $E$ is of cotype 2 these spaces are identical to the space of 2-summing operators from $B$ to $E$.
5) When $E$ is the space $c_0$ these spaces are identical to the space of all operators from $B$ to $E$.

1) If $C_1 \leq C_2$ then $\mathcal{G}(B, E, C_2) \subset \mathcal{G}(B, E, C_1)$ (the order is reverse).
2) $\mathcal{G}(B, E, 1) \subset \mathcal{F}(B, E, 1)$.
3) These spaces contains the space of 2-summing operators.
4) When $E$ is of cotype 2 these spaces are identical to the space of 2-summing operators from $B$ to $E$.
5) When $E$ is the space $c_0$ these spaces are identical to the space of all operators from $B$ to $E$.

1) If $C_1 \leq C_2$ then $\mathcal{G}(B, E, C_2) \subset \mathcal{G}(B, E, C_1)$ (the order is reverse).
2) $\mathcal{G}(B, E, 1) \subset \mathcal{F}(B, E, 1)$.
3) These spaces contains the space of 2-summing operators.
4) When $E$ is of cotype 2 these spaces are identical to the space of 2-summing operators from $B$ to $E$.
5) When $E$ is the space $c_0$ these spaces are identical to the space of all operators from $B$ to $E$.

1) If $C_1 \leq C_2$ then $\mathcal{G}(B, E, C_2) \subset \mathcal{G}(B, E, C_1)$ (the order is reverse).
2) $\mathcal{G}(B, E, 1) \subset \mathcal{F}(B, E, 1)$.
3) These spaces contains the space of 2-summing operators.
4) When $E$ is of cotype 2 these spaces are identical to the space of 2-summing operators from $B$ to $E$.
5) When $E$ is the space $c_0$ these spaces are identical to the space of all operators from $B$ to $E$.

1) If $C_1 \leq C_2$ then $\mathcal{G}(B, E, C_2) \subset \mathcal{G}(B, E, C_1)$ (the order is reverse).
2) $\mathcal{G}(B, E, 1) \subset \mathcal{F}(B, E, 1)$.
3) These spaces contains the space of 2-summing operators.
4) When $E$ is of cotype 2 these spaces are identical to the space of 2-summing operators from $B$ to $E$.
5) When $E$ is the space $c_0$ these spaces are identical to the space of all operators from $B$ to $E$.

We recall a preceding Proposition

### Proposition

*Let $B$ be a subspace of $c_0$ and $E = \mathcal{C}(K)$ where $K$ is a compact Hausdorff space. Then every operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\| \leq \|T\|_1 \leq 2\|T\|$.*

The Problem is the following: Does this proposition holds true when $\mathcal{F}$ is replace by $\mathcal{G}$ ?

We recall a preceding Proposition

**Proposition**

*Let $B$ be a subspace of $c_0$ and $E = \mathcal{C}(K)$ where $K$ is a compact Hausdorff space. Then every operator $T : B \to E$ belongs to $\mathcal{F}(B, E, C)$, for every $C$, and $\|T\| \leq \|T\|_1 \leq 2\|T\|$.*

The Problem is the following: Does this proposition holds true when $\mathcal{F}$ is replace by $\mathcal{G}$ ?