Heat and Navier-Stokes equations in supercritical function spaces

Franka Baaske

Friedrich Schiller University Jena, Germany

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Navier-Stokes equations

We consider

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}, \nabla)\mathbf{u} - \Delta_x \mathbf{u} + \nabla P &= 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty), \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0 \quad \text{in } \mathbb{R}^n, \end{aligned}$$

where $n \geq 2$. Here $\mathbf{u}(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ denotes the unknown velocity vector, $P(x,t)$ the unknown scalar pressure.

$$[(\mathbf{u}, \nabla)\mathbf{u}]^k = \sum_{j=1}^{n} u^j \frac{\partial}{\partial x_j} u^k, \quad k = 1, \ldots, n,$$

$$\operatorname{div} \mathbf{u} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} u^j, \quad \nabla P = (\frac{\partial}{\partial x_1} P, \ldots, \frac{\partial}{\partial x_n} P).$$

Navier-Stokes equations

Reformulation:

$$\frac{\partial}{\partial t}\mathbf{u} - \Delta_x \mathbf{u} + \mathbb{P}\,\mathrm{div}(\mathbf{u}\otimes\mathbf{u}) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0 \quad \text{in } \mathbb{R}^n,$$

based on

$$(\mathbf{u},\nabla)\mathbf{u} = \mathrm{div}(\mathbf{u}\otimes\mathbf{u}) \qquad \mathrm{div}(\mathbf{u}\otimes\mathbf{u})^k = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u^j u^k)$$

and the Leray projector

$$(\mathbb{P}f)^k = f^k + R_k \sum_{j=1}^{n} R_j f^j, \quad k = 1,\dots,n$$

with the scalar Riesz transforms

$$R_k f(x) = i\left(\frac{\xi_k}{|\xi|}\hat{f}\right)^{\vee}(x) = c_n \lim_{\varepsilon\downarrow 0}\int_{|y|\geq\varepsilon} \frac{y_k}{|y|^{n+1}}\, f(x-y)\mathrm{d}y, \quad x\in\mathbb{R}^n.$$

Remark: H. Triebel: Local Function Spaces, Heat and Navier-Stokes Equations

Aim of the talk: Strong solutions of the Navier-Stokes equations, based on the reduction to a nonlinear heat equation in Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$.

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So far it was assumed, that solutions spaces are multiplication algebras.

$$A^s_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n) \text{ with } \begin{cases} s > n/p & \text{where } 0 < p, q \leq \infty, \\ s = n/p & \text{where } 0 < p < \infty,\ 0 < q \leq 1, \end{cases}$$

or

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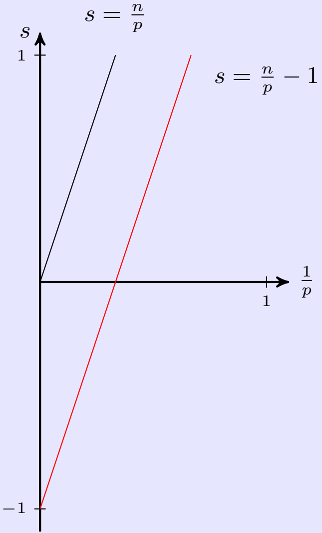
- we deal with $0 < \frac{1}{p} - \frac{s}{n} = \frac{1}{r} < \frac{1}{n}$, $s > 0$, $n \geq 2$
- spaces consist of regular distributions



- $s = \frac{n}{p} - 1$ critical line
- with respect to initial data: $A^{\sigma}_{p,q}$ with $\sigma < \frac{n}{p} - 1$ subcritical not well adapted for initial data
- in particular $L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$, $2 \leq p < n$
- we consider spaces $A^{s-\frac{n}{r}-1+g}_{p,q}$, $s - \frac{n}{r} - 1 + g > \frac{n}{p} - 1$ $\rightsquigarrow$ cover all supercritical spaces in the context of Navier-Stokes equations
- relation between well-posedness and critical spaces: $\mathcal{C}^{-1} = B^{-1}_{\infty,\infty}$ ill-posed and critical

Definition:

Let $\phi_0 \in S(\mathbb{R}^n)$ with

$$\phi_0(x) = 1, \text{ if } |x| \leq 1 \quad \text{and} \quad \phi_0(x) = 0, \text{ if } |x| \geq \frac{3}{2}$$

and put

$$\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}.$$

Then $\sum_{j=0}^{\infty} \phi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

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Definition:

Let $s \in \mathbb{R}$, $0 < p, q \le \infty$, ($p < \infty$ for F-spaces), then

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}\phi_j \mathcal{F} f|L_p(\mathbb{R}^n)\|^q\right)^{1/q},$$

$$\|f|F^s_{p,q}(\mathbb{R}^n)\| = \| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}\phi_j \mathcal{F} f(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\|.$$

Vector-valued version:

$$A^s_{p,q}(\mathbb{R}^n)_n = \prod_{j=1}^{n} A^s_{p,q}(\mathbb{R}^n), \qquad \text{where } A \in \{B, F\}$$

which collects all $f = (f^1, \dots, f^n)$ such that $f^j \in A^s_{p,q}(\mathbb{R}^n)$ and

$$\|f|A^s_{p,q}(\mathbb{R}^n)_n\| = \sum_{j=1}^{n} \|f^j|A^s_{p,q}(\mathbb{R}^n)\|.$$

Solution spaces:

X Banach space with $X \subset S'(\mathbb{R}^n)$, $0 < T < \infty$, $b \in \mathbb{R}$, $1 \le v \le \infty$

Then $L_v((0,T),\, b,\, X) = \{f(\cdot,t) \in X,\, 0 < t \le T : \|f|L_v((0,T),\, b,\, X)\| < \infty\}$ with

$$\|f|L_v((0,T),\, b,\, X)\| = \left(\int_0^T t^{bv} \|f(\cdot,t)|X\|^v \mathrm{d}t\right)^{1/v}$$

(usual modification if $v = \infty$).

$\rightsquigarrow \quad X = A^s_{p,q}(\mathbb{R}^n)$ or $X = A^s_{p,q}(\mathbb{R}^n)_n$ with appropriately chosen parameters s, p, q.

Nonlinear heat equation

Reduction: $\mathbb{P}\,\mathrm{div}(\mathbf{u}\otimes\mathbf{u}) \quad \rightsquigarrow \quad Du^2 = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} u^2$

Nonlinear heat equation as scalar model case of the Navier-Stokes equations:

$$\begin{aligned} \frac{\partial}{\partial t}u - \Delta_x u - Du^2 &= 0, && \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot,0) &= u_0, && \text{in } \mathbb{R}^n. \end{aligned}$$

$\omega \in S'(\mathbb{R}^n)$ a regular distribution, Gauss-Weierstrass semi-group:

$$W_t\,\omega(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \omega(y)\mathrm{d}y, \quad x \in \mathbb{R}^n,\, t > 0.$$

ω, f regular distributions (subject to some restrictions), then its unique solution is given by:

$$W(x,t) = W_t\,\omega(x) + \left(\int_0^t W_{t-\tau} f_\tau \mathrm{d}\tau\right)(x.)$$

Strong solution: $u(x,t)$ solution of a fixed point problem, unique in some Banach space X for $x \in X$, belongs to $C([0,T), X(\mathbb{R}^n))$

Theorem:

Let $n \geq 2$, $1 \leq p, q \leq \infty$, $s \geq 0$, $-1 < s - \frac{n}{p} < 0$

($p < \infty$ for F- spaces, $q \leq r$ for B- spaces) with $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$. Then $r > n$.

Furthermore let $v,\ a,\ g \in \mathbb{R}$ with

$$\frac{1}{2} + \frac{3n}{2r} < g \leq 1 + \frac{n}{r} \quad \text{and} \quad 0 \leq \frac{1}{v} < \frac{1}{2}(1 - \frac{n}{r}),$$
$$a = 1 - \frac{1}{v} + \frac{n}{r} - \lambda \quad \text{with} \quad \frac{1}{2} + \frac{3n}{2r} < \lambda < g$$

and $u_0 \in A_{p,q}^{s-1-\frac{n}{r}+g}(\mathbb{R}^n)$ for the initial data. Then there exists a number $T > 0$ such that

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) - \Delta_x u(x,t) - Du^2(x,t) &= 0, && \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) &= u_0(x), && \text{in } \mathbb{R}^n, \end{aligned}$$

has a unique solution

$$u \in L_{2v}((0,T), \frac{a}{2}, A_{p,q}^{s}(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0,T)) \cap C([0,T), A_{p,q}^{s-1-\frac{n}{r}+g}(\mathbb{R}^n)).$$

Basic ideas of the proof

fixed point problem: $T_{u_0}u(x,t) := W_t u_0(x) + \left(\int_0^t W_{t-\tau} Du^2(x,\tau)\mathrm{d}\tau\right)$

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estimate:

$$\begin{aligned} \|T_{u_0}u(\cdot,t)|A^s_{p,q}(\mathbb{R}^n)\| &\leq c\,t^{-(\frac{1}{2}+\frac{n}{2r})+\frac{g}{2}}\|u_0|A^{s-1-\frac{n}{r}+g}_{p,q}(\mathbb{R}^n)\| \\ &+ c\,t^{1-\frac{1}{v}-\frac{1}{2}-\frac{n}{2r}}\left(\int_0^t \tau^{av}\|u(\cdot,\tau)|A^s_{p,q}(\mathbb{R}^n)\|^{2v}\mathrm{d}\tau\right)^{1/v} \end{aligned}$$

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based on

$$0 < \frac{1}{r} = \frac{1}{p} - \frac{s}{n} < \frac{1}{2} \quad \text{and} \quad \frac{1}{p_r} = \frac{1}{p} + \frac{1}{r}$$

Then (additionally $q \leq r$, if $A = B$)

$$A^s_{p_r,q}(\mathbb{R}^n) \hookrightarrow A^{s-n/r}_{p,q}(\mathbb{R}^n) \quad \text{and} \quad A^s_{p,q}(\mathbb{R}^n) \cdot A^s_{p,q}(\mathbb{R}^n) \hookrightarrow A^s_{p_r,q}(\mathbb{R}^n)$$

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$\rightsquigarrow$ unique solution in $U_T \subset L_{2v}((0,T), \frac{a}{2}, A^s_{p,q}(\mathbb{R}^n))$, extension via iteration with $d = 1 + \frac{n}{r} + \eta$ and embedding $A^{s+k\eta}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{s+k\eta-\frac{n}{p}}(\mathbb{R}^n)$

Theorem:
Let $n \geq 2$, $1 < p < \infty$, $1 < q < \infty$ for F-spaces, $1 \leq q \leq r$ for B-spaces,

$$s \geq 0 \quad , -1 < s - \frac{n}{p} < 0 \text{ with } \frac{1}{r} = \frac{1}{p} - \frac{s}{n}.$$

Furthermore let $v,\ a,\ g \in \mathbb{R}$ with

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and $u_0 \in A_{p,q}^{s-1-\frac{n}{r}+g}(\mathbb{R}^n)_n$ for the initial data. Then there exists a number $T > 0$ such that the Navier-Stokes equations

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u}(x,t) - \Delta_x \mathbf{u}(x,t) + \mathbb{P}(\operatorname{div}(\mathbf{u} \otimes \mathbf{u}))(x,t) &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x) \ \text{ in } \mathbb{R}^n, \end{aligned}$$

have a unique solution

$$\mathbf{u} \in L_{2v}((0,T), \frac{a}{2}, A_{p,q}^{s}(\mathbb{R}^n)_n) \cap C^\infty(\mathbb{R}^n \times (0,T))_n \cap C([0,T), A_{p,q}^{s-1-\frac{n}{r}+g}(\mathbb{R}^n)_n).$$

Basic ideas of the proof

Recall that $(\mathbb{P}f)^k = f^k + R_k \sum_{j=1}^n R_j f^j, \quad k = 1, \dots, n,$ with the scalar Riesz transforms

$$R_k f(x) = i \left(\frac{\xi_k}{|\xi|} \hat{f} \right)^\vee (x) = c_n \lim_{\varepsilon \downarrow 0} \int_{|y| \geq \varepsilon} \frac{y_k}{|y|^{n+1}} f(x-y) \mathrm{d}y, \quad x \in \mathbb{R}^n.$$

mapping properties: R_k generates a linear and bounded map in

$$B^s_{p,q}(\mathbb{R}^n), \quad \text{if } 1 < p < \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R},$$

and in

$$F^s_{p,q}(\mathbb{R}^n), \quad \text{if } 1 < p < \infty,\ 1 < q < \infty,\ s \in \mathbb{R}.$$

$\rightsquigarrow$ reduction to the main assertion (scalar version of the above theorem)

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