# Function Spaces XI

## Paley spaces

Sergey Astashkin (Samara State University)

Joint work with E. Semenov (Voronezh State University)

# Haar functions

Let $\mathbb{D}$ be the family of dyadic intervals $\Delta_n^k = ((k-1)2^{-n}, k2^{-n})$, $k = 1, \dots, 2^n$, $n = 0, 1, \dots$ The functions $h_0^0(t) = 1$,

$$h_n^k(t) = h_{\Delta_n^k}(t) = \begin{cases} 1, t \in \Delta_{n+1}^{2k-1} \\ -1, t \in \Delta_{n+1}^{2k} \\ 0 \text{ for the remaining } t \in [0,1] \end{cases}$$

form the *Haar system*.

Let $\Omega := \{(0,0), (n,k), k = 1, \dots, 2^n, n = 0, 1, \dots\}$. The sequence $\{2^{n/2} h_n^k\}_{(n,k)\in\Omega}$ is a complete orthonormal system and a monotone basis in $L_p[0,1]$ for every $1 \le p < \infty$ (J. Schauder, 1928).

# Paley–Marcinkiewicz theorem

**Definition.**

A basis $\{e_k\}_{k=1}^{\infty}$ of a Banach space $E$ is called *unconditional* if its basis property is preserving under any permutation of its elements or, equivalently, if there is a constant $C > 0$ such that for all $\varepsilon_m = \pm 1$ and $c_m \in \mathbb{R}$

$$\Big\| \sum_{m=1}^{\infty} \varepsilon_m c_m e_m \Big\|_E \le C \Big\| \sum_{m=1}^{\infty} c_m e_m \Big\|_E .$$

**Theorem(J. Marcinkiewicz, 1937, R.E. Paley, 1932)**

For every $1 < p < \infty$ the Haar system is an unconditional basis in $L_p[0,1]$.

At the same time, Haar functions form a conditional basis in $L_1[0,1]$:

$$\Big\| h_0^1 + \sum_{n=1}^{N} 2^{n-1} h_n^1 \Big\|_{L_1} = 1,\ N \in \mathbb{N}, \ \text{ but } \lim_{N\to\infty} \Big\| h_0^1 + \sum_{n=1}^{N} (-1)^n 2^{n-1} h_n^1 \Big\|_{L_1} = \infty.$$

Let the Fourier-Haar coefficients $c_n^k = c_n^k(x)$ of a function $x \in L_1[0,1]$ be defined by

$$c_n^k = 2^n \int_0^1 x(s) h_n^k(s)\, ds, \ \ (n,k) \in \Omega.$$

The square (or *Paley*) function

$$Px(t) := \Big( \sum_{(n,k)\in\Omega} (c_n^k h_n^k(t))^2 \Big)^{1/2},$$

is finite a.e. on $[0,1]$.

**An equivalent formulation of Paley–Marcinkiewicz theorem.**

For every $1 < p < \infty$ there is $A_p > 0$ such that

$$A_p^{-1} \int_0^1 |x(t)|^p\, dt \le \int_0^1 Px(t)^p\, dt \le A_p \int_0^1 |x(t)|^p\, dt.$$

## Symmetric spaces

**Definition.**

A Banach space $X$ of measurable functions on $[0,1]$ is called *symmetric* (or *rearrangement invariant*) if:

**1.** $X$ is an ideal lattice, that is, if $|x(t)| \leqslant |y(t)|$ a.e. on $[0,1]$ and $y \in X$, then $x \in X$ и $\|x\|_X \leqslant \|y\|_X$;

**2.** if $y \in X$ and

$$|\{t : |y(t)| > \tau\}| = |\{t : |x(t)| > \tau\}|$$

for all $\tau > 0$, then $x \in X$ и $\|x\|_X = \|y\|_X$.

## Orlicz spaces

Every increasing convex function $M(u)$, $M(0) = 0$, on $[0, \infty)$ generates the *Orlicz space* $L_M$ with the norm

$$\|x\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M(|x(t)|/\lambda)\,dt \leqslant 1\right\}.$$

If $M(u) = u^p$, then $L_M = L_p$. The Orlicz space $L_M$, where $M(u) = e^{u^2} - 1$ (resp. $M(u) = u\log(1+u)$) will be denoted by $\exp L_2$ (resp. $L\log L$).

## Boyd indices

In any symmetric space $X$, the operator $\sigma_\tau x(t) = x(t/\tau)\chi_{[0,1]}(t/\tau)$, $\tau > 0$, is bounded, and $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$. The *Boyd indices* of $X$:

$$\alpha_X = \lim_{\tau\to 0} \frac{\ln\|\sigma_\tau\|_X}{\ln\tau} \quad\text{and}\quad \beta_X = \lim_{\tau\to\infty} \frac{\ln\|\sigma_\tau\|_X}{\ln\tau}.$$

Always, $0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1$. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ if $1 \le p \le \infty$.

# Köthe duality

Denote by $X'$ the *Köthe dual* space to a symmetric space $X$ with the norm

$$\|x\|_{X'} = \sup_{\|y\|_X \leqslant 1} \int_0^1 x(t)y(t)\,dt.$$

$X'$ is a symmetric space and $X \subset X''$ with $\|f\|_{X''} \leq \|f\|_X$, $f \in X$.

$X = X''$ (isometrically) iff $X$ has the *Fatou property*, that is, if $0 \leq f_n \nearrow f$ a.e. on $[0,1]$ and $\sup_{n\in\mathbf{N}} \|f_n\|_X < \infty$, then $f \in X$ and $\|f_n\|_X \nearrow \|f\|_X$.

A symmetric space $X$ separable iff $X' = X^*$.

Next, we assume that $X$ is separable or has the Fatou property and $\|\chi_{[0,1]}\|_X = 1$.

# Paley function and Paley spaces

The Haar functions form a monotone basis in any separable symmetric space. It is an unconditional basis in $X$ iff $0 < \alpha_X \leqslant \beta_X < 1$ (E. Semenov, 1969; in the case of Orlicz spaces, V. Gaposhkin, 1967). Equivalently, the Haar system is an unconditional basis in a symmetric space $X$ iff there is $C = C(X)$ such that for all $x \in X$

$$C^{-1}\|x\|_X \leq \|Px\|_X \leq C\|x\|_X. \qquad (1)$$

($Px$ is the Paley function, corresponding to the Haar system).

Given a symmetric space $X$, the *Paley space* $P(X)$ will be defined as the space of all functions $x \in L_1[0,1]$ such that $Px \in X$ with the norm $\|x\|_{P(X)} := \|Px\|_X$. In the case $X = L_1$ we obtain the dyadic $H^1$ space, which belongs to the classes of atomic and martingale $H^1$ spaces (P.Müller, "Isomorphisms between $H^1$ spaces", 2005).

# The operator $P$ in symmetric spaces.

Consider the embeddings $P(X) \subset X$ and $X \subset P(X)$ separately.

**Theorem 1**

Let $X$ be a symmetric space on $[0,1]$.
1. The embedding $P(X) \subset X$ holds iff $\alpha_X > 0$.
2. The embedding $X \subset P(X)$ holds iff $0 < \alpha_X \leqslant \beta_X < 1$.

**Corollary 1.**

If $X \subset P(X)$, then $X = P(X)$.

**Corollary 2: A refined version of criterion of unconditionality of Haar basis.**

The Haar system is an unconditional basis in a separable symmetric space $X$ iff the operator $P$ is bounded in $X$, i.e., there is $C = C(X)$ such that for all $x \in X$

$$\|Px\|_X \le C\|x\|_X.$$

# The Haar system as an RUC-basis

## Definition

Let $X$ be a Banach space. A biorthogonal system $(e_k, e_k^*)$, $e_k \in X$, $e_k^* \in X^*$, is called an *RUC (random unconditional convergence)–system* if, for any (some) $p \in [1, \infty)$, there is $K_p > 0$ such that the inequality

$$\sup_{n=1,2,\ldots} \left( \int_0^1 \Big\| \sum_{k=1}^n r_k(s) e_k^*(x) e_k \Big\|_X^p \, ds \right)^{1/p} \le K_p \|x\|_X$$

holds for any $x \in [e_k]$ (see P. Billard, S. Kwapieǹ, A. Pełczyǹski, Ch. Samuel, Lonhorn Notes. Texas Func. Anal. Seminar 1985-1986).

A basis that (together with the basis coefficient functionals) forms an RUC-system, is called an *RUC-basis*. Clearly, each unconditional basis is an RUC-basis. On the other hand, trigonometric system is an RUC-basis in $L_p[0,1]$ if $p \ge 2$, but it is a conditional basis in $L_p[0,1]$ for all $p \neq 2$.

## Corollary 3.

The Haar system is an RUC-basis in a separable symmetric space $X$ iff it is unconditional (equivalently, $0 < \alpha_X \le \beta_X < 1$).

## Proof

If the Haar system $\{h_k\}_{k=1}^{\infty}$ forms an RUC-basis in $X$, then for some $K$

$$\int_0^1 \Big\| \sum_{k=1}^n r_k(s) c_k h_k \Big\|_X \, ds \le K \Big\| \sum_{k=1}^n c_k h_k \Big\|_X, \quad n \in \mathbb{N}.$$

Moreover, by the $L_1$-Khinchine inequality, $\int_0^1 \| \sum_{k=1}^n r_k(s) c_k h_k \|_X \, ds \ge$

$$\Big\| \int_0^1 \Big| \sum_{k=1}^n r_k(s) c_k h_k \Big| \, ds \Big\|_X \ge \frac{1}{\sqrt{2}} \Big\| \Big( \sum_{k=1}^n (c_k h_k)^2 \Big)^{1/2} \Big\|_X.$$

Consequently, since $X$ is separable, $\|Px\|_X \le \sqrt{2} K \|x\|_X$ for all $x \in X$, and from Theorem 1 it follows that $0 < \alpha_X \le \beta_X < 1$.

# The ideal and symmetric properties of Paley spaces.

**Theorem 2.**

Let $X$ be a symmetric space on $[0,1]$.

1. The following conditions are equivalent:
(a) $P(X)$ is symmetric;
(b) $P(X)$ has the ideal property;
(c) $X = L_2$ isometrically.

2. The following conditions are equivalent:
(a) $P(X)$ coincides (up to equivalence of norms) with some symmetric space;
(b) $P(X)$ coincides (up to equivalence of norms) with some Banach function lattice;
(c) $0 < \alpha_X \leqslant \beta_X < 1$.

# Banach properties of Paley spaces

**Proposition 1.**

For every symmetric space $X$ the space $P(X)$ is complete.

**Proposition 2.**

The space $P(X)$ is separable iff so is $X$.

**Theorem 3.**

If a separable symmetric space $X$ contains a subspace isomorphic to $l_1$ (resp. to $c_0$), then $P(X)$ also contains a subspace isomorphic to $l_1$ (resp. to $c_0$).

**Theorem 4.**

The Paley space $P(X)$ is reflexive iff so is $X$.

# The Rademacher functions in Paley spaces

$$r_n(t) := \mathrm{sign}\,\sin(2^n\pi t), \quad n = 1, 2, \dots$$

If $a = (a_k)_{k=1}^\infty \in \ell_2$, the series $\sum_{n=1}^{\infty} a_n r_n(t)$ converges a.e. on $[0,1]$. Let *Rad* denote the class of all such functions.

Since $r_{n+1}(t) = \sum_{k=1}^{2^n} h_n^k(t)$, $n = 0, 1, 2 \dots$, then $P\left(\sum_{n=1}^\infty a_n r_n\right)(t) = \left(\sum_{n=1}^\infty a_n^2\right)^{1/2}$. So, $Rad \subset P(X)$ and

$$\left\|\sum_{n=1}^{\infty} a_n r_n\right\|_{P(X)} = \|(a_n)\|_{\ell_2}$$

for any symmetric space $X$.

# Complementability of *Rad* in Paley spaces

**Definition**

A closed subspace $F$ of a Banach space $E$ is called complemented (1-complemented) in $E$ if there exists a bounded projection (a norm-one projection) from $E$ onto $F$.

**Theorem 5.**

The subspace $Rad$ is 1-complemented in $P(X)$ for any symmetric space $X$.

More precisely, the orthogonal projection

$$Qf(s) := \sum_{n=1}^{\infty} \int_0^1 f r_n \, dt \cdot r_n(s)$$

is bounded on $P(X)$ and $\|Qf\|_{P(X)} \leq \|f\|_{P(X)}$.

## Corollary 4.

For any symmetric space $X$ the Paley space $P(X)$ contains a 1-complemented subspace isometric to $\ell_2$.

## Comparison with symmetric spaces.

The projection $Q$ from Theorem 5 is bounded in a symmetric space $X$ (equivalently, $Rad$ is complemented in $X$) iff $G \subset X \subset G'$, where $G$ is the closure of $L_\infty$ in the exponential Orlicz space $\exp L_2$, and $G'$ is the Köthe dual space to $G$ (Rodin-Semenov and independently Lindenstrauss-Tzafriri result, 1979). Moreover,the last embeddings are equivalent to the condition that a separable symmetric space $X$ which fails to contain a complemented sublattice isomorphic to $\ell_2$ contains $\ell_2$ as a complemented subspace (Raynaud, 1995).

# Interpolation properties of the Banach couple $(L_\infty, P(L_\infty))$.

Let $(X_0, X_1)$ be a Banach couple, that is, Banach spaces $X_0$ and $X_1$ embedded into some Hausdorff topological linear space; let $x \in X_0 + X_1$, $t > 0$. The functional

$$\mathcal{K}(t, x, X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

is called the Peetre $\mathcal{K}$-*functional* of the couple $(X_0, X_1)$.

**Definition.**

Let $Y_0$ and $Y_1$ be closed subspaces of $X_0$ and $X_1$, respectively. A couple $(Y_0, Y_1)$ is called a $\mathcal{K}$-*closed subcouple* of the couple $(X_0, X_1)$ if

$$\mathcal{K}(t, y, Y_0, Y_1) \leq C\mathcal{K}(t, y, X_0, X_1),$$

where $C$ is independent of $t > 0$ и $y \in Y_0 + Y_1$.

The operator

$$Ra(t) = \sum_{k=1}^{\infty} a_k r_k(t), \quad a = (a_k)_{k=1}^{\infty} \in \ell_2,$$

is an isometry from $\ell_1$ into $L_\infty$ and from $\ell_2$ into $P(L_\infty)$. Therefore, the couples $(R(\ell_1), R(\ell_2))$ and $(\ell_1, \ell_2)$ are isometric and

$$\mathcal{K}(t, Ra; R(\ell_1), R(\ell_2)) = \mathcal{K}(t, a; \ell_1, \ell_2).$$

**Theorem 6.**

The Banach couple $(R(\ell_1), R(\ell_2))$ is a $\mathcal{K}$-closed subcouple of the couple $(L_\infty, P(L_\infty))$.

Let $(X_0, X_1)$ be a Banach couple and $F$ be a Banach lattice on $(0, \infty)$ with measure $dt/t$ such that $\min(1, t) \in F$. *The space of the $\mathcal{K}$-method* $(X_0, X_1)_F^{\mathcal{K}}$ consists of all $x \in X_0 + X_1$ such that $\|x\| := \|\mathcal{K}(t, x, X_0, X_1)\|_F < \infty$.
It is an interpolation space with respect to the Banach couple $(X_0, X_1)$, i.e., an arbitrary linear operator that is bounded on $X_0$ and $X_1$ is also bounded on $(X_0, X_1)_F^{\mathcal{K}}$. If $F$ is a weighted space $L_p(t^{-\theta}, dt/t)$, $0 < \theta < 1, 1 \leq p \leq \infty$, we obtain the classical spaces $(X_0, X_1)_{\theta,p}$.

## Corollary 5.

Let $\tilde{F} = (\ell_1, \ell_2)_F^{\mathcal{K}}$. If $X = (L_\infty, P(L_\infty))_F^{\mathcal{K}}$, then $\|\sum_{n=1}^{\infty} a_n r_n\|_X \asymp \|(a_n)\|_{\tilde{F}}$.

Given arbitrary interpolation space $\tilde{F}$ with respect to the couple $(\ell_1, \ell_2)$, Corollary 5 makes it possible to find an interpolation space $X$ with respect to the couple $(L_\infty, P(L_\infty))$ such that on $Rad \cap X$ the norm of $X$ is equivalent to the norm of $\tilde{F}$.

# An example.

Let $1 < p < 2$. Then $\ell_p = (\ell_1, \ell_2)_{\theta,p}$, where $\theta = 2(p-1)/p$. Therefore, if $X_p := (L_\infty, P(L_\infty))_{\theta,p}$, then

$$\Big\| \sum_{n=1}^{\infty} a_n r_n \Big\|_{X_p} \asymp \|(a_n)\|_{\ell_p}.$$

A closed result (Nikishin,1974;Rodin–Semenov,1975): if $\Lambda_p$ is the Lorentz space with the norm $\|x\|_{\Lambda_p} := (\int_0^1 (x^*(t))^p \ln^{-p}(e/s)\, ds/s)^{1/p}$, then

$$\Big\| \sum_{n=1}^{\infty} a_n r_n \Big\|_{\Lambda_p} \asymp \|(a_n)\|_{\ell_p}.$$

Observe that $\Lambda_p = (L_\infty, G)_{\theta,p}$, where $\theta = 2(p-1)/p$.

# $P(L_\infty)$ as an endpoint of the $L_p$-scale.

For all $p \geq 2$

$$\frac{\sqrt{2}}{\sqrt{p}}\|Pf\|_{L_p} \leq \|f\|_{L_p} \leq 16\sqrt{p}\|Pf\|_{L_p} \qquad (2)$$

with the correct asymptotic behavior of the constants as $p \to \infty$ (see Müller's book "Isomorphisms between $H^1$ spaces", 2005). So, $P(L_\infty)$ appears as an endpoint of the $L_p$-scale for $p \to \infty$.

From (2) it follows that $L_\infty \subset P(\exp L_2)$. Moreover, the function

$$x(t) = \sum_{n=0}^{\infty}(h^1_{2n}(t) - h^1_{2n+1}(t)) \in L_\infty$$

and $Px(t) \geq \log_2^{1/2}(1/t)$, $0 < t \leq 1$.

**Theorem 7.**

Let $X$ be a symmetric space on $[0,1]$. The following conditions are equivalent:

(i) $L_\infty \subset P(X)$;

(ii) $\sup\limits_{E\subset[0,1]} \|\chi_E\|_{P(X)} < \infty$;

(iii) $\exp L_2 \subset X$.

Also, from (2) it follows that $P(L_\infty) \subset \exp L_2$.

The function $\log_2^{1/2}(2/t)$ does not belong to $P(L_\infty)$. However, the following result shows that the exponential Orlicz space $\exp L_2$ is the smallest symmetric space containing $P(L_\infty)$.

## Theorem 8.

For any $x \in \exp L_2$ there exists a function $f \in P(L_\infty)$ such that

$$\frac{1}{C}\|x\|_{\exp L_2} \leqslant \|f\|_{P(L_\infty)} \leqslant C\|x\|_{\exp L_2},$$

where $C > 0$ is independent of $x$, and

$$f^*(t) \geq x^*(t), \ \ 0 < t \leq 1/2.$$

In particular, there exists $f_0 \in P(L_\infty)$, $\|f_0\|_{P(L_\infty)} = 1$, such that

$$|f_0(t)| \leqslant \log_2^{1/2}(2/t), \ \ 0 < t \leq 1,$$

and

$$f_0^*(t) \geqslant \frac{1}{2}\log_2^{1/2}(2/t), \ \ 0 < t \leq 1/2.$$

# Duality for Paley spaces.

From the famous Fefferman's inequality (1972) it follows that $P(L_1)^* = (H^1)^* = BMO_d$, where the dyadic $BMO_d$ space consists of all functions $f \in L_1[0,1]$ such that

$$\|f\|_{BMO_d} := \sup_{I \in \mathbb{D}} \frac{1}{|I|} \int_I |f(s) - f_I| \, ds < \infty,$$

where the supremum is taken over all $I \in \mathbb{D}$, $f_I = \frac{1}{|I|} \int_I f(s) \, ds$.

Let $f \in L_1[0,1]$, $f \sim \sum_{L \in \mathbb{D}} b_L h_L$. Then

$$\|f\|_{BMO_d} \asymp \|f^\sharp\|_{L_\infty}, \quad \text{where} \quad f^\sharp(x) := \sup_{I \in \mathbb{D}:\, x \in I} \left( \frac{1}{|I|} \sum_{L \in \mathbb{D}:\, L \subset I} b_L^2 |L| \right)^{1/2},$$

$0 \le x \le 1$.

Let $X$ be a symmetric space on $[0,1]$. Denote by $X^\sharp$ the space of all $f \in L_1[0,1]$, $f \sim \sum_{L \in \mathbb{D}} b_L h_L$, such that $f^\sharp \in X$, with the norm $\|f\|_{X^\sharp} := \|f^\sharp\|_X$.

**Theorem 9.**

Suppose $X$ is a separable symmetric space such that $\alpha_X > 1/2$. Then $P(X)^* = (X')^\sharp$.

Problem: To find the largest symmetric space embedded into $P(L_1)$.

**Theorem 10.**

A symmetric space $X$ is embedded into $P(L_1)$ iff $X \subset L \log L$.

# When the spaces $P(X)$ and $X$ are isomorphic?

**Conjecture.**

Let $X$ be a symmetric space on $[0,1]$. If the spaces $P(X)$ and $X$ are isomorphic, then $0 < \alpha_X \leqslant \beta_X < 1$.

Let $P(X)$ and $X$ be isomorphic. Since $P(X)$ contains a complemented copy of $\ell_2$, we have $G \subset X \subset G'$. Moreover, if $X$ is a separable space, then the Haar functions form in $P(X)$ an unconditional basis. So, $0 < \alpha_X \leqslant \beta_X < 1$.

Non-separable case ?

Thank you for your attention!