Banach spaces with prescribed ultrapowers

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In Banach space theory (and in other fields of functional analysis), the ultrapower and ultraproduct constructions proved themselves to be fruitful. Recall for instance that given a Banach space X and an ultrafilter $\mathcal{U}$ on an index set I, the $\mathcal{U}$-ultrapower of X, denoted by $X_{\mathcal{U}}$ is an isometric extension of X obtained by quotienting the Banach space $\ell_\infty(I;X)$ by the closed linear subspace consisting of families with limit 0 with respect to $\mathcal{U}$. It is well known in Banach space theory that the space $X_{\mathcal{U}}$ is finitely representable in X, that is all its finite dimensional subspaces are linearly isomorphic, with arbitrarily small distortion, to subspaces of X; however the relationship of $X_{\mathcal{U}}$ to X is far closer than finite representability, as will be explained in the talk.

Two Banach spaces X, Y are said to be *elementarily equivalent* if for some ultrafilter $\mathcal{U}$ their respective ultrapowers $X_{\mathcal{U}}$ and $Y_{\mathcal{U}}$ are linearly isometric. The elementary class of X consists of all Banach spaces that are elementarily equivalent to X.

The talk will aim essentially at two goals, linked by the common loose question: knowing $X_{\mathcal{U}}$, can we recognize X?

– to give classes $\mathcal{C}$ of classical Banach spaces for which X belongs to $\mathcal{C}$ whenever $X_{\mathcal{U}}$ does. The classical results concern L_p and $C(K)$-spaces, but some progress has been made in more recent time that I will summarize briefly.

– to provide non trivial examples of separable Banach spaces the elementary class of which contains exactly one element of density character κ for each uncountable cardinal κ (a trivial example is the Hilbert space). One kind of these examples consists simply of certain Nakano spaces of sequences. This second part reports on a work in progress with C. W. Henson.