Approximation numbers of weighted Sobolev embeddings via bracketing

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Abstract

Let B be the unit ball in $\mathbb{R}^n, m \in \mathbb{N}$ and $1 \leq p < \infty$. We define the weighted Sobolev space $E^m_{p,\sigma}(B)$ as the completion of $C^m_0(B) = \{f \in C^m(B) : \operatorname{supp} f \text{ compact}\}$ with respect to the norm

$$\|f \;|E^m_{p,\sigma}(B)\| := \left(\int_B |x|^{mp} \; (1 + |\log|x||)^{\sigma p} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p}.$$

Then, if $\sigma > 0$, the embedding

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$$

is compact. In case of Hilbert spaces, $p = 2$, Triebel obtained in [3] sharp results for the corresponding approximation numbers

$$a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & , \text{if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} & , \text{if } \sigma = \frac{m}{n} \\ k^{-\sigma} & , \text{if } 0 < \sigma < \frac{m}{n}. \end{cases}$$

Therefor the Courant-Weyl method of Dirichlet-Neumann-bracketing was used. This technique is not available for $p \neq 2$, but a partial analogue was established by Evans and Harris in [1] for Sobolev spaces $W^1_p(\Omega)$ on a wide class of domains. We want to transfer this idea and extend the results to the general case of Banach spaces $1 < p < \infty$.

References

[1] W.D. Evans, D.J. Harris. Fractals, trees and the Neumann Laplacian. *Math. Ann.*, 296: 493-512, 1993.

[2] T. Mieth, Entropy and approximation numbers of embeddings of weighted Sobolev spaces. *J. Approx. Theory*, 192: 250-272, 2015.

[3] H. Triebel. Entropy and approximation numbers of limiting embeddings, an approach via Hardy inequalities and quadratic forms. *J. Approx. Theory*, 164(1): 31-46, 2012.