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Structure of Cesàro function spaces and interpolation

Abstract

The *Cesàro function spaces* $Ces_p(I)$ on both $I = [0,1]$ and $I = [0,\infty)$ are classes of Lebesgue measurable real functions f on I such that the norm

$$\|f\|_{C(p)} = \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx\right]^{1/p} < \infty, \quad \text{for } 1 \le p < \infty,$$

and

$$\|f\|_{C(\infty)} = \sup_{x\in I,\ x>0} \frac{1}{x}\int_0^x |f(t)|\, dt < \infty, \quad \text{for } p = \infty.$$

In the case $1 < p < \infty$ spaces $Ces_p(I)$ are separable, strictly convex and not symmetric. They, in the contrast to the sequence spaces, are not reflexive and do not have the fixed point property.

The structure of the Cesàro function spaces $Ces_p(I)$ was investigated in [1]–[3] and [6]–[7]. Their dual spaces, which equivalent norms have different description on $[0,1]$ and $[0,\infty)$, are described. The spaces $Ces_p(I)$ for $1 < p < \infty$ are not isomorphic to any $L^q(I)$ space with $1 \le q \le \infty$. They have "near zero" complemented subspaces isomorphic to l^p and "in the middle" contain an asymptotically isometric copy of l^1 and also a copy of $L^1[0,1]$. They do not have Dunford-Pettis property. Cesàro function spaces on $[0,1]$ and $[0,\infty)$ are isomorphic for $1 < p < \infty$. Moreover, the Rademacher functions span in $Ces_p[0,1]$ for $1 \le p < \infty$ a space which is isomorphic to l^2. This subspace is uncomplemented in $Ces_p[0,1]$. The span in the space $Ces_\infty[0,1]$ gives another sequence space.

In [5] and [8] it was shown that $Ces_p(I)$ is an interpolation space between $Ces_{p_0}(I)$ and $Ces_{p_1}(I)$ for $1 < p_0 < p_1 \le \infty$, where $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$. The same result is true for Cesàro sequence spaces. On the other hand, $Ces_p[0,1]$ is not an interpolation space between $Ces_1[0,1]$ and $Ces_\infty[0,1]$.

More general spaces were considered in [10]–[12]. For a Banach ideal function space X on I we define the *abstract Cesàro space* $CX = CX(I)$, the *abstract Copson space* $C^*X = C^*X(I)$ and the *abstract Tandori space* $\widetilde{X} = \widetilde{X}(I)$ as

$$CX = \{f \in L^0(I) : C|f| \in X\} \quad \text{with the norm } \|f\|_{CX} = \|\, C|f|\,\|_X,$$

$$C^*X = \{f \in L^0(I) : C^*|f| \in X\} \quad \text{with the norm } \|f\|_{C^*X} = \|\, C^*|f|\,\|_X,$$

$$\widetilde{X} = \{f \in L^0(I) : \widetilde{f} \in X\} \quad \text{with the norm } \|f\|_{\widetilde{X}} = \|\,\widetilde{f}\,\|_X,$$

where $Cf(x) = \frac{1}{x}\int_0^x f(t)\,dt$, $C^*f(x) = \int_{I\cap[x,\infty)} \frac{f(t)}{t}\,dt$ and $\widetilde{f}(x) = \operatorname{ess\,sup}_{t\in I,\, t\ge x} |f(t)|$.

Comparisons of Cesàro, Copson and Tandori spaces as well as the "iterated" spaces CCX and C^*C^*X are presented in [12]. It may happen that spaces are different but the corresponding Cesàro, Copson and Tandori spaces are the same, that is, there are $X \neq Y$ such that $CX = CY$, $C^*X = C^*Y$ and $\widetilde{X} = \widetilde{Y}$.

The duality of abstract Cesàro spaces was proved in [10]: under some mild assumptions on X we have $(CX)' = \widetilde{X'}$ in the case $I = [0, \infty)$ and $(CX)' = \widetilde{X'(1/v)}$, where $v(x) = 1 - x,\ x \in [0, 1)$ in the case $I = [0, 1]$

The real and complex interpolation methods of abstract Cesàro, Copson and Tandori spaces, including the description of Calderón-Lozanovskiĭ construction for those spaces were given in [11].

The investigations show an interesting phenomenon that there is a big difference between properties and interpolation of Cesàro function spaces in the cases of finite and infinite interval.

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