PARANORMED GENERALIZATIONS OF L^p SPACES

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Abstract

Given a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the set of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. For a bijection $\varphi : (0, \infty) \to (0, \infty)$ define $\mathbf{P}_\varphi : S \to [0, \infty)$,

$$\mathbf{P}_\varphi(x) := \begin{cases} \varphi^{-1}\left(\int_{\Omega(x)} \varphi \circ |x| \, d\mu\right) & \text{if } \mu(\Omega(x)) > 0 \\ 0 & \text{if } \mu(\Omega(x)) = 0 \end{cases},$$

where $\Omega(x)$ is the support of $x \in S$.

The conditions under which the functional $\mathbf{P}_\varphi$ is a paranorm (F-norm) in $S(\Omega, \Sigma, \mu)$ are considered. In particular the bijections φ and ψ such that each of the following conditions is satisfied:

1) $\mathbf{P}_\varphi$ is is subhomogeneous, i.e.

$$\mathbf{P}_\varphi(tx) \le t\mathbf{P}_\varphi(x), \qquad t > 1, \ x \in S;$$

2) $\mathbf{P}_\varphi$ is subadditive, i.e.

$$\mathbf{P}_\varphi(x + y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \qquad x, y \in S(\Omega, \Sigma, \mu);$$

3) the Hölder-type inequality holds true,

$$\int_\Omega xy d\mu \le \mathbf{P}_\varphi(x)\mathbf{P}_\psi(y), \qquad x, y \in S(\Omega, \Sigma, \mu).$$

are characterized.

If the measure space is such that there are two sets $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty, \tag{*}$$

the first of these inequalities implies that φ is a power function, the second that $\varphi(t) = \varphi(1)\, t^p$ for some $p \ge 1$, and the Hölder-type inequality implies that φ and ψ are conjugate power functions.

In connection with the Hölder-type inequality, three kinds of conjugate functions, (generalizing the power conjugate functions) are introduced, and the relevant questions are considered.